P.B. SIDDHARTHA COLLEGE OF ARTS & SCIENCE

(Autonomous)

SIDDHARTHA NAGAR, VIJAYAWADA – 520 010

A COLLEGE WITH POTENTIAL EXCELLENCE

ISO 9001:2015 NAAC Accredited

DEPARTMENT OF PHYSICS

WAVESANDOSCILLATIONS

CourseCode:23PHMAL122

CourseCode:23PHMAL122 **Offeredto:B.Sc.(H)**

DomainSubject:PHYSICS Semester–II

Max.Marks:100(CIA:30+SEE:70) TheoryHrs./Week:3

Credits:04

Unit	LearningUnits	Lecture Hours
I Simple Harmonic	**A)** Simple harmonic oscillator and solution of the differential equation- Physical characteristics of SHM, Torsion pendulum-measurements of rigidity modulus, Compound pendulum- measurement of 'g', **B)** Principle of superposition, the combination of two mutually perpendicularsimpleharmonicvibrationsofthesamefrequencyand differentfrequencies, –applicationsLissajousfigures	12
IIDamped and forced	**A)** Simple harmonic oscillator, damped harmonic oscillator, Logarithmic decrement, Relaxation time, and Quality factor. **B)** forcedharmonicoscillator-differentialequationsanditssolutions Resonance-amplituderesonance.	12
IIIComple x	**A)** Fourier theorem (Statement & limitations), evaluation of the Fourier coefficients using Fourier's theorem **B)** Analysisofperiodicwavefunctions-squarewave,Saw tooth wave.	12
IVVibratingStrings and Bars	**A)** Transverse wave propagation along a stretched string, Velocity of a transverse wave along a stretched string, modes of vibration of stretched string clamped at ends, overtones and harmonics. **B)** GeneralsolutionoftheLongitudinalwaveequation.Specialcases (i) bar fixed at both ends (ii) bar fixed at the midpoint (iii) bar fixed at one end.	12

vUltrasonics,	Ultrasonics, properties of ultrasonic waves, production of Ultrasonics by piezo-electricandmagnetostrictionmethods,detectionofUltrasonics, Applicationsandusesofultrasonicwaves.	12

Text Book

1. *B.Sc.Physics,Vol.1,TeluguAcademy,Hyderabad*

Model Question Paper

WavesandOscillations

SECTION-A

Answerthefollowing: 5x10=50M

1. *A) Define simple harmonic motion. Derive the equation of a simple harmonic oscillator and obtain its solution (L3. CO1)*

(OR)

B) *Discuss the combination of two mutually perpendicular simple harmonic vibrations (L3, CO1)*

2. *A) What are damped oscillations? Derive the equation of motion of a forced oscillator and find its solution (L3, CO3)*

(OR)

B) *What are forced oscillations? Derive the equation of motion of a forced oscillator and obtain its solution (L3, CO2)*

3 A)StateFourier'stheoremandevaluatetheFouriercoefficients.(L3,CO3).

(OR)

B)AnalyseasquarewaveusingtheFouriertheorem.(L3,CO3)

4.A) Derive an expression for the velocity of a transverse wave along a stretched string.(L3, CO4).

(OR)

B) Deduce the modes of vibration of a rod clamped at one end and free at the other end (L2, CO4)

5 A)DescribetheMagnetostrictionmethodofproducingultrasonicwaves.(L2,CO5)

(OR)

B)DescribethePiezo-electricmethodofproducingultrasonicwaves(L2,CO5)

SECTION-B

Answerany**THREE**ofthefollowingquestions: 3x4=12M

6. *Explainbrieflythephysicalcharacteristicsofsimpleharmonicmotion(L1,CO1)*
7. *Definerelaxationtimeanddriveanexpressionforit.(L2,CO2)*
8. *MentionthelimitationsofFourier'stheorem(L1,CO3)*
9. *Explainovertonesandharmonics.(L1,CO4)*
10. *WriteanyfiveapplicationsofUltrasonics.(L1,CO5)*

Section–C 2X4=8M

Answerany**TWO**ofthefollowing:

9. *Aspringofforceconstant20NM^{-1}isloadedwithamassof0.1kgandallowedto oscillate. Calculate the time period and frequency of oscillation of the string (L4, CO1)*
10. *The amplitude of an oscillator of frequency 200Hz falls to 1/10th of its initial value after a time of 10s. Calculate its relaxation time and Q-factor. (L4, CO2)*
11. *A steel wire of length 50cm has a mass of 5gm. It is stretched with a tension of 400N.Calculate the frequency of the wire in the fundamental mode of vibration (L3, CO4)*
12. *Calculate the fundamental frequency of a quartz crystal of thickness 0.003m given Y-8X10^{10}Pa and density is 2500kgm^{-3} for quartz (L3, CO5)*

WAVESANDOSCILLATIONS

PRACTICALS

CourseCode:23PHMAP122 Offeredto:B.Sc.(H)

DomainSubject:PHYSICS Semester–II

Max. Marks:50(CIA:15+SEE:35) TheoryHrs./Week:2

Credits:01

COURSEOBJECTIVE:

Todeveloppracticalskillsintheuseoflaboratoryequipmentandexperimental techniques for measuring properties of matter and analyzing mechanical systems

Courseoutcomes: Onsuccessfulcompletionofthiscourse,thestudentswillbeableto:

CO 1 Gainhands-onexperienceinsettingupandconductingexperimentsrelatedto waves and oscillations.

CO 2 Investigateandanalyzethebehaviorofdifferenttypesofwaves,suchas mechanical waves, sound waves, and electromagnetic waves.

CO 3 Examine resonance phenomena in various systems and understand the conditionsthat lead to resonance.

CO 4 Enhance skills in presenting findings through graphical representations and written reports.

CO 5 Develop critical thinking skills by solving problems related to wave mechanics and oscillatory systems.

ListofExperiments

1. *Volumeresonatorexperiment*
2. *Determinationof'g'bycompound/barpendulum*
3. *Simple pendulum normal distribution of errors-estimation of time period and the error of the mean by statistical analysis*
4. *Determinationoftheforceconstantofaspringbystaticanddynamicmethods.*
5. *Determinationoftheelasticconstantsofthematerialofaflatspiralspring.*
6. *Coupledoscillators*
7. *Verificationoflawsofvibrationsofstretchedstring–sonometer*
8. *Determinationoffrequencyofabar–Melde'sexperiment.*
9. *Study of a damped oscillation using the torsional pendulum immersed in liquid-decay constant and damping correction of the amplitude.*
10. *FormationofLissajousfiguresusingCRO.*

Note:

1. *8 (Eight) Experiments are to be done and recorded in the lab. These experiments will be evaluated by the CIA.*
2. *For certification minimum of 6 (Six) experiments must be done and recorded by*

students who had put in 75 % of attendance in the lab.

3. ***Thebest6experimentsaretobeconsideredfortheCIA*.**
4. *10+5(RECORD)=**15**marksforCIA*
5. ***35**marksforthepracticalexam.*

ThemarksdistributionfortheSemesterEndpracticalexaminationisasfollows:

Formula/Principle/Statementwithanexplanationofsymbols	05
Diagram/CircuitDiagram/TabularColumns	05
Settingupoftheexperimentandtakingreadings/Observations	10
Calculations(explicitlyshown)+Graph+ResultwithUnits	05
ProcedureandPrecautions	04
Result	01
Viva-voce	05
TotalMarks:	35

UNIT-I
FUNDEMENTALS OF VIBRATION

SIMPLE HARMONIC MOTION (SHM):

The acceleration of a body in periodic motion along a straight line is directly proportional to its displacement but in opposite direction and is always directed towards a fixed point, then the body is said to be in simple harmonic motion.

Properties:

1. The motion is periodic.
2. The motion is along a straight line about the mean position.
3. The acceleration is directly proportional to its displacement but in opposite direction
4. Acceleration is always directed towards its mean position.

Ex: Simple pendulum, vibration of prongs of a tuning fork etc.

THE SIMPLE OSCILLATOR:

When a particle or a body moves such that its acceleration is always directed towards a fixed point and varies directly as its distance from that point, the particle or body is said to execute S.H.M. The particle or body executing simple harmonic motion is called a Simple oscillator.

Equation of motion of simple oscillator:

Consider a particle 'P' of mass 'm' executing SHM about an equilibrium position 'O' along X- axis as shown in figure.

```
-----------------•---------------
                 O       P
                 |       |---x-- -|
```

By definition, the restoring force is directly proportional to the displacement (x) but in opposite direction.

i.e., $F \propto -x$ or $F = -kx$ -------- (1)

Where k = proportionality constant or force constant

= force per unit displacement

'–' ve sign indicates 'F' and 'x' are in opposite direction.

According to Newton's-II Law of motion, the restoring force on mass m produces an acceleration, $a = \frac{d^2x}{dt^2}$ on the mass, so, that

F = mass x acceleration, i.e.,   F = m a i.e.,   F = m $\frac{d^2x}{dt^2}$ **------** (2)

From equations (1) & (2) we get,

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \textbf{--------} \ (3)$$

Where, $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$

Eq. (3) is known as differential equation of simple harmonic oscillator.

SOLUTION OF DIFFERENTIAL EQUATION OF SIMPLE OSCILLATOR:

Let, $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d\upsilon}{dt} \because \frac{dx}{dt} = \upsilon$

$$= \frac{d\upsilon}{dx} \cdot \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \upsilon \cdot \frac{d\upsilon}{dx} \quad \textbf{-----------------} \ \textbf{(4)}$$

The equation of motion of Simple harmonic oscillator is,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

From eq. (4) $\upsilon \cdot \frac{d\upsilon}{dx} = -\omega^2 x$

$$\upsilon \, d\upsilon = -\omega^2 x \, dx$$

On *Integrating,* $\int \upsilon \, d\upsilon = -\omega^2 \int x \, dx$

$$\frac{\upsilon^2}{2} = \frac{-\omega^2 x^2}{2} + C_1,$$ Where C_1= Integrating constant

The value of C_1 is calculated by applying the condition at x =a (amplitude) velocity of the particle is zero ($\upsilon = 0$)

$$\therefore \quad 0 = \frac{-\omega^2 a^2}{2} + C_1$$

$$\therefore C_1 = \frac{\omega^2 a^2}{2}$$

$$\therefore \frac{\upsilon^2}{2} = \frac{-\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$$

$$\upsilon^2 = \omega^2 (a^2 - x^2)$$

$$\upsilon = \omega\sqrt{(a^2 - x^2)} \text{ ----------------- (5)}$$

As$\upsilon = \frac{dx}{dt}$, eq (5) is written as

$$\omega\sqrt{(a^2 - x^2)} = \frac{dx}{dt}$$

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = \omega\, dt \text{ -------------------------------(6)}$$

To integrate eq. (6) substitute x = a Sinθ. Hence, dx = a Cosθdθ

$$\frac{a\, Cos\theta\, d\theta}{\sqrt{(a^2 - a^2 Sin^2\theta)}} = \omega\, dt$$

$$\frac{a cos\theta d\theta}{a cos\theta} = \omega\, dt$$

$$d\theta = \omega\, dt \text{ -----------------(7)}$$

Integrating eq. (7), we get $\theta = (\omega t + \emptyset)$, where $\emptyset$ is a constant

Now, the displacement $x = a\, Sin\, (\omega t + \phi)$ -----------------(8)

This is the displacement of the particle at any instant.

If the motion of the particle is on Y-axis,

$$y = a\cos(\omega t + \phi) \text{ --------------(9)}$$

CHARACTERISTICS OF SHM

1. **Displacement (x):** The displacement of any particle at any instant executing SHM is given by

$$x = a\sin(\omega t + \phi)$$

The maximum displacement from mean position is called amplitude.

Here, amplitude = a

2. **Velocity (υ):** The velocity of the oscillating particle is given by

$$\upsilon = \frac{dx}{dt} = a\,\omega\cos(\omega t + \phi)$$

$$\upsilon = a\omega\sqrt{1 - \sin^2(\omega t + \phi)}$$

$$\upsilon = \omega\sqrt{a^2 - a^2\sin^2(\omega t + \phi)}$$

$$\upsilon = \omega\sqrt{(a^2 - x^2)}$$

At mean position, $x = 0$, $\upsilon = \omega a$ is maximum

i.e., $\upsilon_{max} = \omega a$

At extreme position, $x = a$, $\upsilon = 0$

3. Time Period (T): time taken for one complete oscillation is called time period.

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\sqrt{\frac{d^2x/dt^2}{x}}} \quad \because \frac{d^2x}{dt^2} = -\omega^2 x$$

$$T = 2\pi\sqrt{\frac{x}{d^2x/dt^2}}$$

$$T = 2\pi\sqrt{\frac{displacement}{acceleration}}$$

4. Frequency (ν): The number of oscillations made in one second is called frequency. $\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \because \omega = \sqrt{\frac{k}{m}}$

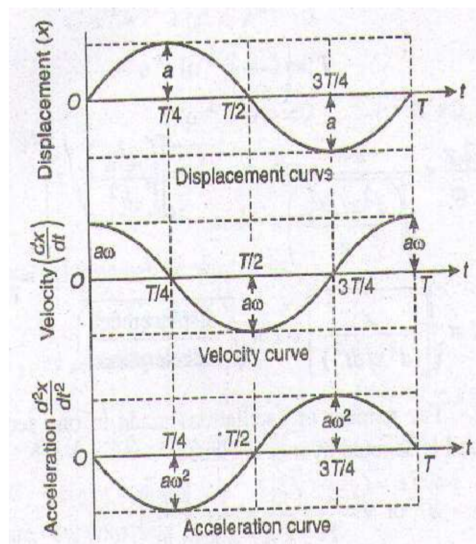
$$\nu = \frac{1}{2\pi}\sqrt{\frac{acceleration}{displacement}}$$

5. **Phase**: Phase denote the position and direction of the particle at any instant of time. The angle, $(\omega t + \phi)$ is called phase of vibration.

6. Epoch: The value of phase when t = 0 is called the initial phase (or) epoch.

Here, ϕ is called epoch.

Relation between displacement, velocity and acceleration:



The displacement of the particle executing SHM is given by, $x = a \sin(\omega t + \phi)$

Its velocity, $\upsilon = \frac{dx}{dt} = a\,\omega \cos(\omega t + \phi)$

Its acceleration, $\frac{d^2x}{dt^2} = -a\omega^2 \sin(\omega t + \phi)$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

If $\phi = 0$, $x = a \sin \omega t = a \sin\left(\frac{2\pi t}{T}\right) \because$

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

$$\upsilon = a\,\omega \cos\left(\frac{2\pi t}{T}\right)$$

$$\text{Acceleration} = -a\,\omega^2 \sin\left(\frac{2\pi t}{T}\right)$$

TORSIONAL PENDULUM-MEASUREMENTS OF RIGIDITY MODULUS:

Torsional Pendulum consists of a heavy metal sphere or cylinder suspended from a rigid support by means of experimental wire. When the sphere or cylinder is slightly twisted in the horizontal plane and the released, the pendulum starts torsional oscillations about the axis of suspension.

Theory:

Let a sphere or cylinder of mass M be suspended at one end of a wire of length l and radius r keeping its other end fixed at a rigid support.

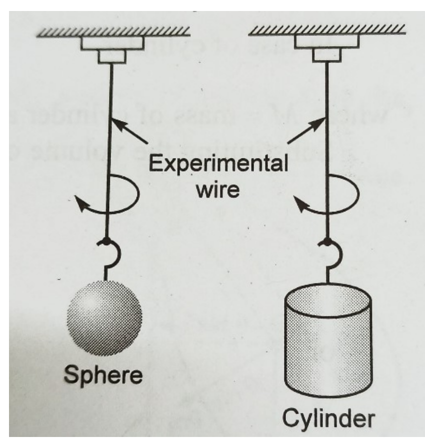
Let, a pendulum be slightly twisted in the horizontal plane through an angle θ radians and then released. The pendulum starts executing torsional oscillations. Let I be the moment of inertia of cylinder or sphere about the axis of suspension.

Within the elastic limits, the couple or torque acting on the wire is proportional to the displacement.

Therefore, $\tau = I\alpha$,

Where angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$ and internal couple acting, $\tau = I\frac{d^2\theta}{dt^2}$.

If C be the torsional rigidity of suspension wire (i.e., couple required to produce unit radian twist in the wire), the restoring couple (τ) required to produce θ radians is $-C\theta$.



In equilibrium, $$I\frac{d^2\theta}{dt^2} = -C\theta. \text{---------------}(1)$$

Therefore, the equation of motion of the pendulum is,

$$I\frac{d^2\theta}{dt^2} + C\theta = 0 \text{ or } \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

$$\textbf{or } \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{where, } \omega^2 = \frac{C}{I} \text{-------}(2)$$

This is the differential eq. of simple harmonic motion

whose time period T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{C}{I}}} \text{ ------------}(3)$$

We know that torsional rigidity C of a wire is given by $C = \frac{\pi\eta\ ^4}{2l}$ ---------(4)

Where η is the modulus of rigidity of the material of wire and I is the moment of inertia.

In case of sphere, $I = \frac{2}{5}MR^2$,

Where M= mass of the sphere and R= radius of sphere.

In case of cylinder, $I = \frac{1}{2}MR^2$,

Where M= mass of the cylinder and R= radius of cylinder.

Substituting the value of C from eq, (4) in eq (3), we get.

$$T = 2\pi\sqrt{\left[\frac{I}{\frac{\pi\eta r^4}{2l}}\right]} = 2\pi\sqrt{\left[\frac{2Il}{\pi\eta r^4}\right]}$$

Or $\quad T^2 = \frac{8\pi^2 Il}{\pi\eta r^4} = \frac{8\pi I}{\eta r^4}$

$$\therefore \boldsymbol{\eta = \frac{8\pi^2 Il}{T^2 r^4}}$$

Measurement of Rigidity Modulus By Torsional Pendulum:

The following procedure is adopted:

(i) The sphere or the cylinder is suspended from a rigid support with the help of experimental wire.

(ii) The sphere or the cylinder is slightly rotated about the wire and released so that it begins to execute torsional oscillations of small amplitude about the wire as axis.

(iii) Start stop watch and simultaneously count the number of oscillations. The time period is $\quad \mathbf{T}=\frac{\textit{total number of oscillations}}{\textit{total time taken}}$

(iv) Measure the length *l* and radius *r* of the wire. The radius of the wire is measured with the help of screw guage and length *l* with the help of meter scale.

(v) With the help of Vernier Callipers measures the radius R of the sphere or cylinder.

(vi) Measure the mass M (in Kg) of the (sphere or cylinder) with the help of physical balance.

Calculate $\quad I = \frac{2}{5}M R^2$ (for sphere)

$I = \frac{1}{2} MR^2$ (for cylinder)

Using the formula $\boldsymbol{\eta = \frac{8\pi^2 Il}{T^2 r^4}}$, we calculate the rigidity modulus of the wire.

Therefore, **For cylinder,** $\boldsymbol{\eta = \frac{8\pi^2 l}{T^2 r^4}\cdot\left(\frac{1}{2} MR^2\right) = \frac{4\pi MR^2 l}{T^2 r^4}}$.

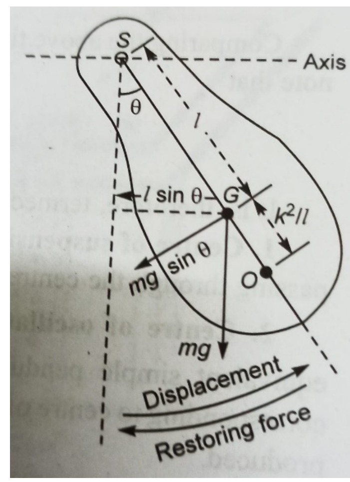
For sphere, $\boldsymbol{\eta = \frac{8\pi^2 l}{T^2 r^4}\cdot\left(\frac{2}{5} MR^2\right) = \frac{16\pi MR^2 l}{5\, T^2 r^4}}$.

COMPOUND PENDULUM:

A compound pendulum is a rigid body, capable of oscillating about a horizontal axis passing through it (not through its centre of gravity) in a vertical plane.

Consider the vertical section of an irregular rigid body pivoted at a point S. In the equilibrium position of the body, the centre of mass lies vertically below S. Let *m*be the mass of the body and *l* the distance between the point of suspension *S* and centre of gravity G.

Let, at any instant *t*, the body be displaced through an angle θ. Let a restoring couple acts on the body to bring it in its mean position of the rest. Due to inertia, it does not stop in the position of rest but swings to opposite side, i.e., the body executes simple harmonic motion.



Theory:

The time period is calculated as follows,

Resorting couple = weight x perpendicular distance of *G* from *S*

$\therefore \tau = mg \times l \sin\theta$

or $\tau = mg\, l.\theta$ ($\because \sin\theta = \theta$, when θ is small).

If I is the moment of inertia of the body about an

axis through *S* perpendicular to the plane of oscillation,

and $\frac{d^2\theta}{dt^2}$ angular acceleration, the torque acting on it $\tau = I\frac{d^2\theta}{dt^2}$

and thus $I\frac{d^2\theta}{dt^2} = -mg\, l\,.\,\theta$

negative sign indicates that angular acceleration is always towards the position of rest.

Then,

$\frac{d^2\theta}{dt^2} = -\frac{mg\, l}{I}\theta = -p^2\theta$ Where, $\frac{mg\, l}{I} = p^2$

This is the equation of simple harmonic motion whose time period T is given as,

$$\mathbf{T} = \frac{2\pi}{p} = 2\pi\sqrt{\left(\frac{I}{mgl}\right)} \quad \text{-----------------------(A)}$$

If I_g be the moment of inertia of the body about its centre of gravity, then from the theorem of parallel axes

$$I = I_g = ml^2$$

or $I = mk^2 + ml^2$----------------------------(B)

where k is the radius of gyration about an axis through the centre of gravity.

Substituting the value of *Ifrom* eq. (B) into eq. (A)

$$T = 2\pi\sqrt{\left(\frac{mk^2 + ml^2}{mgl}\right)} = 2\pi\sqrt{\left(\frac{\frac{k^2}{l}+l}{g}\right)}$$

Comparing the above time period with the periodic time of the simple pendulum

$$T=2\pi\sqrt{\left(\frac{L}{g}\right)}, \text{ we get } L = \frac{k^2}{l} + l$$

It is, therefore, termed as the length of the equivalent simple pendulum.

PRINCIPLE OF SUPERPOSITION OF WAVES

According to the principle of superposition, when a medium is distributed simultaneously by any number of waves, the instantaneous resultant displacement of the medium at every instant is the algebraic sum of the displacements of the medium due to individual waves in absence of others.

If $y_1, y_2, y_3, \ldots$ be the displacement vectors due to waves 1,2, 3,.. acting separately, then the resultant displacement is

$$y= y_1+y_2+y_3+\ldots\ldots$$

The following are the important cases of the superposition of waves:

(i) Two waves of the same frequency moving in the same direction **(Interference of waves).**

(ii) Two waves of the slightly different frequencies moving in the same direction **(Beats).**

(iii) Two waves of the same frequency moving in the opposite direction **(Stationary Waves).**

COMBINATION OF TWO MUTUALLY PERPENDICULAR SIMPLE HARMONIC VIBRATIONS

EQUAL FREQUENCIES

Let us consider two simple harmonic motions having the same frequency one acting along X-axis and the other acting along Y-axis. Let the two vibrations be represented by

$x = a \,\mathrm{Sin}\,(\omega t + \phi)$ ---------- (1)

and $y = b \,\mathrm{Sin}\,\omega t$ ----------- (2)

where a, b arethe amplitudes of ‘x’ and ‘y’ vibrations respectively.

The *x* motion is ahead of the *y* motion by angle ϕ i.e., the phase different between the two vibrations is ϕ.

The equation of resultant vibrations is obtained by eliminating t between eqs. (1) and (2)

From eq. (2), $\mathrm{Sin}\,\omega t = \left(\frac{y}{b}\right)$

$\therefore \quad \mathrm{Cos}\,\omega t = \sqrt{1 - Sin^2\omega\, t} = \sqrt{1 - \frac{y^2}{b^2}}$

Expanding eq (1) and substituting the values of sin ωt and cos ωt, we get

From eq. (1), $\frac{x}{a} = \mathrm{Sin}\,\omega t \,\mathrm{Cos}\,\phi + \mathrm{Cos}\,\omega t\,\mathrm{Sin}\,\phi$

$$\frac{x}{a} = \frac{y}{b}\,\mathrm{Cos}\,\phi + \sqrt{1 - \frac{y^2}{b^2}}\,\mathrm{Sin}\,\phi$$

$$\frac{x}{a} - \frac{y}{b}\,\mathrm{Cos}\,\phi = \sqrt{1 - \frac{y^2}{b^2}}\,\mathrm{Sin}\,\phi$$

Squaring on both sides,

$$\left(\frac{x}{a} - \frac{y}{b}Cos\phi\right)^2 = \left(1 - \frac{y^2}{b^2}\right)\mathrm{Sin}^2\,\phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\mathrm{Cos}^2\,\phi - \frac{2xy}{ab}\mathrm{Cos}\,\phi = \mathrm{Sin}^2\,\phi - \frac{y^2}{b^2}\mathrm{Sin}^2\,\phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}(\mathrm{Cos}^2\,\phi + \mathrm{Sin}^2\,\phi) - \frac{2xy}{ab}\mathrm{Cos}\,\phi = \mathrm{Sin}^2\,\phi$$

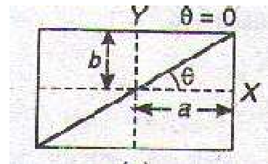
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi = \sin^2 \phi \quad \text{---------} \ (3)$$

This equation represents oblique ellipse, which is the resultant path of the particle.

Special Cases:

1. when $\phi = 0$ (i.e., the two vibrations are in phase)

Cos $\phi = 1$ and sin $\phi = 0$



From Eq. (3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$

fig (i)

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$\pm\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

$$\therefore \pm \quad y = \pm \frac{b}{a} x \ \text{---------} \ (4)$$

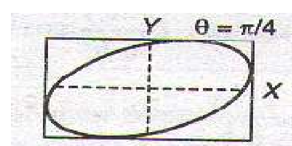
This represents two coincident straight lines passing through the origin and inclined to X-axis at an angle ‘θ’.

$$\text{Tan } \theta = \frac{b}{a} \quad \text{(or)} \ \theta = Tan^{-1}\left(\frac{b}{a}\right)$$

This resultant path is shown in fig. (i)

2. When $\phi = \frac{\pi}{4}$ we have,

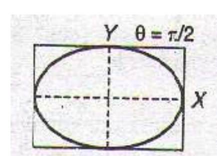
Cos $\phi = \frac{1}{\sqrt{2}}$ and Sin $\phi = \frac{1}{\sqrt{2}}$



From Eq. (3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \frac{1}{\sqrt{2}} = \frac{1}{2}$ --------- (5)

This represents an oblique ellipse, shown in the figure.

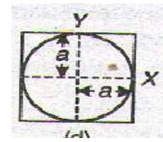
3.When $\phi = \frac{\pi}{2}$ we have,



Cos $\phi = 0$ and Sin $\phi = 1$

From Eq. (3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ --------- (6)

The resultant path is an ellipse, whose major axis coincides with the coordinate axis as shown in fig.

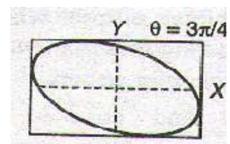


If a = b , then $x^2 + y^2 = a^2$

So, the resultant path of the particle is a circle of radius 'a' as shown in figure.

4.When $\phi = \frac{3\pi}{4}$ we have,

$\cos\phi = -\frac{1}{\sqrt{2}}$ and $\sin\phi = \frac{1}{\sqrt{2}}$



From Eq.(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\sqrt{2}\ xy}{ab} = \frac{1}{2}$ --------- (7)

This equation represents an oblique ellipse, as shown in figure

5.when $\phi = \pi$. We have, $\cos\phi = -1$ and $\sin\phi = 0$

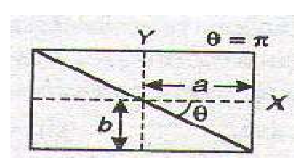
From Eq. (3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$

$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$

$\pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0 \Rightarrow \pm\frac{x}{a} = \pm\frac{y}{b}$

$\therefore \pm\ y = \pm\ \frac{b}{a}x$ --------- (8)

This again represents two coincident straight lines passing through the origin and inclined to X-axis at an angle 'θ'.



$\tan\theta = -\frac{b}{a}$ (or) $\theta = Tan^{-1}\left(-\frac{b}{a}\right)$

This resultant path is shown in figure.

DIFFERENT FREQUENCIES (FREQUENCIES IN THE RATIO 1:2

Consider two simple harmonic motions have the same frequency in the ratio 2:1 one acting along X-axis and the other acting along Y-axis. These vibrations are represented by

x = a Sin (2ωt + ϕ) ---------- (1)

and y = b Sin ωt ----------- (2)

where a, b are their respective amplitudes and ϕ is the phase angle by which x-vibration the initially ahead of y-vibration. The equation of the resultant vibration is obtained by eliminating *t* between eqs. (1) & (2)

From eq. (2), $\text{Sin }\omega t = \left(\frac{y}{b}\right)$, $\text{Cos}\omega t = \sqrt{1 - sin^2\omega t}$

$$\therefore \quad \cos\omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

Expanding Eq. (1) we get,

$$\frac{x}{a} = \text{Sin } 2\omega t \text{ Cos } \phi + \text{Cos } 2\omega t \text{ Sin } \phi$$

$$\frac{x}{a} = 2 \sin \omega t \cos \omega t \cos \phi + (1\text{-}2 \text{ Sin}^2\omega t) \sin \phi$$

Substituting the value of sin ωt and cos ωt, we have

$$\frac{x}{a} = \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}} \text{ Cos } \phi + \left(1 - \frac{2y^2}{b^2}\right)\sin \phi$$

$$\text{or} \quad \frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right)\sin \phi = \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}} \cos \phi$$

Squaring both sides

$$\frac{x^2}{a^2} + \left(1 - \frac{2y^2}{b^2}\right)^2 \sin^2\phi - \frac{2x}{a}\left(1 - \frac{2y^2}{b}\right)\sin \phi = \frac{4y^2}{b^2}\left(1 - \frac{y^2}{b^2}\right)\cos^2 \phi$$

$$\frac{x^2}{a^2} + \sin^2\phi + \frac{4y^4}{b^4} \sin^2\phi - \frac{4y^2}{b^2} \sin^2\phi - \frac{2x}{a} \sin \phi + \frac{4\,x\,y^2}{a\,b^2}\sin \phi = \frac{4y^2}{b^2}\cos^2 \phi - \frac{4\ y^4}{b^4} \cos \phi$$

$$\frac{x^2}{a^2} + \sin^2\phi - \frac{2x}{a} \sin \phi + \frac{4\ y^4}{b^4} (\sin^2\phi + \cos^2 \phi) - \frac{4y^2}{b^2}(\sin^2\phi + \cos^2 \phi) + \frac{4\,x\,y^2}{a\,b^2}\sin \phi = 0$$

$$\left(\frac{x}{a} - \sin\emptyset\right)^2 + \frac{4\ y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4\,x\,y^2}{a\,b^2}\sin \phi = 0$$

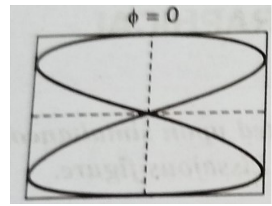
$$\left(\frac{x}{a} - \sin\emptyset\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}\sin\phi - 1\right) = 0 \text{---------------------(3)}$$

This is the equation of a curve having two loops, which is the resultant path.

Special Case

(i) When ϕ=0, π, 2π, When the two component vibrations are in phase.

Substituting ϕ=0 in eq. (3),

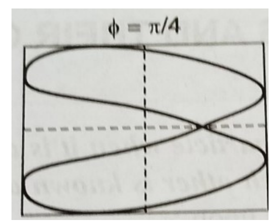


we have, $\left(\frac{x^2}{a^2}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0$ -----------------(4)

This is represented in the figure.

(ii) When $\phi=\frac{\pi}{4}$.

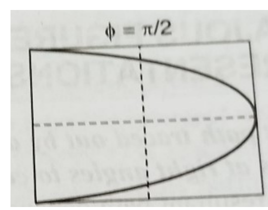
In this case $sin\emptyset = \frac{1}{\sqrt{2}}$. The eq. (3) is



$$\left(\frac{x^2}{a^2} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{a\sqrt{2}}\right) = 0 \text{-----------(5)}$$

This represents a curve as shown in the figure.

(iii) *When* $\phi=\frac{\pi}{2}$, we have $sin\emptyset = 1$. Then eq. (3) gives,



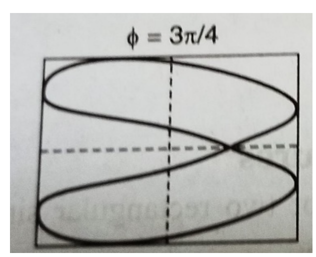
$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4\,y^4}{b^4} + \frac{4y^2}{b^2}\left(\frac{x}{a} - 1\right) = 0$$

$$\left\{\left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2}\right\} = 0$$

This represents two coincident parabolas, the equation of each parabola being

$$\left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2} = 0 \quad \text{or} \quad \frac{2y^2}{b^2} = -\left(\frac{x}{a} - 1\right)$$

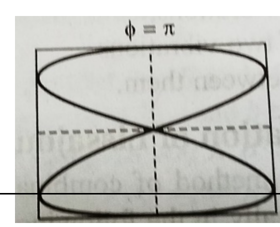


∴ $y^2 = -\frac{b^2}{2a}(x - a)$ --------------(6)

The pair of coincident parabolas symmetrical about x-axis is shown in the figure.

(iv) When $\phi=\frac{3\pi}{2}$. In this case $sin\emptyset = \frac{1}{\sqrt{2}}$. The eq. (3) reduces to the same form as in case (ii).

Hence the path of resultant vibration is the same.



(v) When $\phi=\pi$. In this case $\sin\phi=0$. Hence, the figure is again obtained as shown in the figure.

LISSAJOUS FIGURES

The resultant path traced out by a particle when it is acted upon simultaneously by two simple harmonic motions at right angles to each other is known as Lissajous figure.

The nature of resultant path depends on,

(i) The amplitude of vibrations

(ii) The frequencies of two vibrations

(iii) The phase difference between them.

USES OF LISSAJOUS FIGURES

1. The ratio of the frequencies of two vibrating systems can be obtained from their Lissajous figure provided the ratio is in a whole number i.e., 1:1, 1:2, 1:3, …so. on
2. The Lissajous figure provide a good method for adjusting the frequencies of two forks to a given ratio.
3. Lissajous figures may be used to determine the frequency of a tuning fork provided the frequency of the other tuning fork producing the figure is known and are comparable i.e., in a whole number ratio.
4. These figures are useful in testing the accuracy of a tuning of some simple intervals between two forks.
5. The figures may be employed to investigate how the period of a rod fixed at one end varies with the length of the rod.
6. Helmholtz used these figures to investigate the variation of a violin string.

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Problems:

1. A particle executing SHM has a maximum velocity of 0.4 m/s and a maximum acceleration of 0.8 m/s^2. Calculate the amplitude and the period of oscillation.

Sol: υ_{max}=a ω = 0.4 m/s

a_{max}= a ω^2 = 0.8 m/s^2

$$\frac{a_{max}}{\upsilon_{max}} = \frac{a\omega^2}{a\omega} = \omega = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{T} = \frac{0.8}{0.4} = 2$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi = 3.14 \text{ sec}$$

υ_{max}=a ω (or) amplitude, $a = \frac{\upsilon_{max}}{\omega} = \frac{0.4}{2} = 0.2\,m$

2.The displacement of a particle executing SHM is

x = 0.01 Sin 100π (t +0.005) m.

Calculate amplitude, periodic time, maximum velocity and displacement at the time of start?

Sol: Given that, x = 0.01 Sin 100π (t +0.005) m

x = 0.01 Sin (100π t +0.5π) m

The general equation is, x = a Sin (ωt + ϕ)

On comparison we get,

(i) amplitude a = 0.01 m and ω = 100π

(ii) Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{100\pi} = 0.02$ Sec

(iii) υ_{max}=a ω = 0.01X 100π = π = 3.14 m/s

(iv) displacement at the time of start (t = 0)

x = 0.01 Sin 100π (0.005)

x = 0.01 Sin π/2

x = 0.01 m

3. A particle executing SHM makes 100 complete oscillations per minute and its maximum speed is 5 m/s. what is the length of its path and maximum acceleration?

Find the velocity when the particle is half wave between its mean position and the extreme position?

Sol: $\nu = \frac{100}{60} = \frac{10}{6}$

$\omega = 2\pi\nu = \frac{2\pi \times 10}{6} = \frac{20 \times 3.14}{6} = 10.47$ rad/s

$\upsilon_{max} = a\,\omega = 5$ m/s

amplitude, $a = \frac{\upsilon_{max}}{\omega} = \frac{5}{10.47} = 0.48\ m$

length of the path = 2 a = 2 x 0.48 = 0.96 m

$a_{max} = a\,\omega^2 = 0.48 \times (10.47)^2 = 52.62$ m/s^2

The velocity of the particle, $\upsilon = \omega\sqrt{(a^2 - x^2)}$

$\upsilon = \omega\sqrt{a^2 - \frac{a^2}{4}} \quad = \omega\sqrt{\frac{3a^2}{4}}$ at the half wave, x = a/2

$\upsilon = \frac{\sqrt{3}\,\omega a}{2} = \frac{1.732 \times 10.47 \times 0.48}{2} = 4.352$ m/s

UNIT-II

DAMPED AND FORCED OSCILLATION

Free Vibrations

When a body is capable of vibrations is displaced from its mean position of equilibrium and then released, it begins to vibrate. In an ideal harmonic oscillator, the amplitude of vibration remains constant for an infinite time, such vibrations are called ***free vibrations*** and the frequency of vibration is called as ***natural frequency.***

Damped Vibrations

The vibrations of a freely vibrating body (such as a pendulum or spring) gradually diminish in amplitude and ultimately die away, as the oscillating system is always subjected to frictional forces arising from air resistance, such vibrations are known as damped vibrations.

Forced Vibrations

When a body is made to vibrates by an external periodic force (which may or may not have its frequency equal to the natural frequency of the body), the body starts vibrating with its own natural frequency but ultimately it vibrates with the frequency of applied force, such vibrations are called *forced vibrations* The forces vibrations, after removal of external periodic force, become free and die out in course of time.

DAMPED HARMONIC OSCILLATOR

In an ideal harmonic oscillator, the amplitude of vibration remains constant for an infinite time. When a body vibrates in air or in any medium which offers resistance to its motion, the amplitude of vibration decreases gradually and ultimately the body comes to rest i.e., the body is subjected to frictional forces arising from air resistance and the motion of the body is known as damped simple harmonic motion.

Examples:

1. If we displace a pendulum from its equilibrium position it will oscillate with a decreasing amplitude and finally come to rest in equilibrium position.

2. Let a mass m is suspended from the spring and set to vibrate. The mass vibrates for a longer time in air as compared to the mass which vibrates partially in air and partially in liquid kept below the mass. The damped force is more when the mass moves in the liquid and hence the vibrations die out more quickly in the liquid than in air.

DIFFERENTIAL EQUATION OF MOTION OF DAMPED HARMONIC OSCILLATOR

There are two opposing forces acting on the damped oscillator,

1. The restoring force (f_1) is directly proportional to the displacement (x) but in opposite direction.

i.e., $f_1 \alpha - x$ (or) $f_1 = -\mu x$

where μ = proportionality constant (or) force constant i.e., force per unit displacement

2. A frictional force (f_2) proportional to velocity (υ) but in opposite direction

i.e., $f_2 \alpha -\upsilon$ (or) $f_2 \alpha - \frac{dx}{dt} \because \frac{dx}{dt} = \upsilon$

(or) $f_2 = -r \frac{dx}{dt}$

where r = frictional force per unit velocity

$\therefore$ The resultant force, F = $f_1 + f_2$

$$F = -\mu x - r \frac{dx}{dt}$$

But, F = m a where, m = mass of the particle

$F = m \frac{d^2x}{dt^2} \because a = \frac{d^2x}{dt^2}$

$\therefore$ Equation of the motion of the particle is, $m \frac{d^2x}{dt^2} = -\mu x - r \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} + \frac{r}{m} \frac{dx}{dt} + \frac{\mu}{m} x = 0$$

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega^2 x = 0 \quad \text{-------- (1)}$$

This is known as differential equation of damped harmonic oscillator.

where, $\frac{r}{m} = 2b$ Here, b = damping constant

$\frac{1}{b}$ = decay modulus

$$\omega^2 = \frac{\mu}{m} \text{ (or) } \omega = \sqrt{\frac{\mu}{m}}$$

SOLUTION OF THE EQUATION FOR VARIOUS BOUNDARY CONDITIONS

Equation (1) is a second-degree differential equation.

Let its solution be, $x = A e^{\alpha t}$--------(2)

where A, α are arbitrary constants

Differentiating eq.(2) with respect to t, we get,

$$\frac{dx}{dt} = A \alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = A \alpha^2 e^{\alpha t}$$

Substituting these values in eq. (1) we get,

$$A \alpha^2 e^{\alpha t} + 2b A \alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

$$A e^{\alpha t} (\alpha^2 + 2b\alpha + \omega^2) = 0$$

$$A e^{\alpha t} \neq 0, \quad \therefore \alpha^2 + 2b\alpha + \omega^2 = 0$$

$$\therefore \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = 2b, c = \omega^2$$

$$\alpha = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}$$

The general solution of equation (1) is

$$x = A_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right) t} + A_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right) t} \text{--------(3)}$$

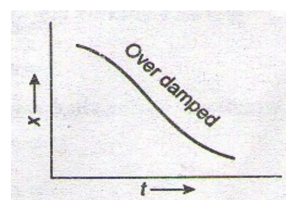
where A_1, A_2 are arbitrary constants.

Special Cases – Different Damping Conditions

Case (1): Over damped motion:

When $b^2 > \omega^2$. In this case $\sqrt{b^2 - \omega^2}$ is real and less than 'b'

Hence, ($-b + \sqrt{b^2 - \omega^2}$) and ($-b - \sqrt{b^2 - \omega^2}$) are both negative.



Thus, the two-displacement x consists of two terms, both dying off exponentially to zero without performing any oscillations, as shown in figure.

The rate of decrease of displacement is governed by the term ($-b + \sqrt{b^2 - \omega^2}$)t as the other term reduces to zero.

In this case, the body once displaced returns to its equilibrium position quite slowly without performing any oscillation, this type of motion is called over damped (or) dead beat.

Ex: 1. Pendulum moving in thick oil.

2. Dead beat moving coil galvanometer.

Case (2): Critical damping:

When $b^2 = \omega^2$. By substituting $b^2 = \omega^2$, the solution does not satisfy eq.(1)

Let us consider, $\sqrt{b^2 - \omega^2} \neq 0$ but, equal to very small quantity 'h'

i.e., $\sqrt{b^2 - \omega^2} = h \rightarrow 0$

From eq. (3), $x = A_1 e^{(-b+h)\,t} + A_2 e^{(-b-h)\,t}$

$$x = e^{-bt}(A_1 e^{ht} + A_2 e^{-ht})$$

$$x = e^{-bt}[A_1(1 + ht + \ldots) + A_2(1 - ht + \ldots)]$$

$$x = e^{-bt}[(A_1 + A_2) + ht(A_1 - A_2) + \ldots]$$

$$x = e^{-bt}[p + q\,t] \quad \text{--------(4)}$$

Where, $p = (A_1 + A_2)$ and $q = h(A_1 - A_2)$

This is a possible form of solution.

From eq. (4), as 't' increases the factor (p + q t) increases, but the factor e^{-bt} decreases.So, the displacement (x) first increases, due to the factor (p + q t) and approaches to zerodue to e^{-bt} as 't' increases.

In this case the particle tends to acquire equilibrium position much rapidly than case (1), this motion is called critical damping.

Ex: This type of motion is exhibited by many pointer instruments such as Ammeter, Voltmeter, etc., in which the pointer moves to the correct position and comes to rest without any oscillations in the minimum time.

<u>Case (3):</u> Under damped motion

When $b^2 < \omega^2$. In this case $\sqrt{b^2 - \omega^2}$ is imaginary

Let, $\sqrt{b^2 - \omega^2} = i \sqrt{\omega^2 - b^2} = i\,\beta$

Where, $i^2 = -1$ (or) $i = \sqrt{-1}$ and $\beta = \sqrt{\omega^2 - b^2}$

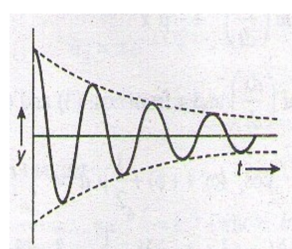
From eq. (3), $x = A_1 e^{(-b + i\beta)\,t} + A_2 e^{(-b - i\beta)\,t}$

$$x = e^{-bt}(A_1 e^{i\beta t} + A_2 e^{-i\beta t})$$

$$x = e^{-bt}[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)]$$

$$x = e^{-bt}[(A_1 + A_2)\cos\beta t + i(A_1 - A_2)\sin\beta t]$$

$$x = e^{-bt}[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t]$$



where, $a\sin\phi = (A_1 + A_2)$ $\quad a\cos\phi = i(A_1 - A_2)$

$$\therefore\ x = e^{-bt}\,a\sin(\beta t + \phi)$$

$$x = a\,e^{-bt}\sin[(\sqrt{\omega^2 - b^2})t + \phi] \text{--------(5)}$$

This is in Simple Harmonic Motion with amplitude ' $a\,e^{-bt}$.

and Time period $T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega^2 - b^2}}$

The amplitude is continuously decreasing due to 'e^{-bt}', where, e^{-bt} is called damping factor.

As $\sin[(\sqrt{\omega^2 - b^2})t + \phi]$ varies between +1 and –1, the amplitude also varies between $a e^{-bt}$ and $-a e^{-bt}$,

The decay of amplitude depends on damping coefficient 'b'. It is called under damped motion as shown in figure.

The time period is slightly increased or frequency decreased because the period is $\frac{2\pi}{\sqrt{\omega^2 - b^2}}$, while in the absence of damping it was $\frac{2\pi}{\omega}$.

Ex: Motion of a pendulum in air, motion of coil of ballistic galvanometer or the electric oscillations of L-C-R circuit.

LOGARITHMIC DECREMENT:

Logarithmic decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

Logarithmic decrement measures the rate at which the amplitude dies away.

The amplitude of damped harmonic oscillator = $a e^{-bt}$

At $t = 0$, amplitude $a_0 = a$

Let a_1, a_2, a_3….. be the amplitudes at time t = T, 2T,3T,…respectively, where T = time period of oscillation.

Then $a_1 = a e^{-bT}$

$a_2 = a e^{-b(2T)}$

$a_3 = a e^{-b(3T)}$ ……….

From these equations, we get

$$\therefore \frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots\ldots = e^{bT} = e^{\lambda},$$ Where $bT = \lambda$ = logarithmic decrement

Taking natural logarithm, we get

$$\boldsymbol{\lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} = \ldots\ldots(1)}$$

RELAXATION TIME (T)

The Relaxation Time (T) is defined as the time taken for the total mechanical energy to decay to (1/e) of its original value.

The mechanical energy of damped oscillator, $E = \frac{1}{2} a^2\mu\, e^{-2bt}$

Let $E = E_0$ when t = 0, $E_0 = \frac{1}{2} a^2\mu$ ---------- (1)

Now, $E = E_0\, e^{-2bt}$ ---------- (2)

Let τ be the relaxation time, $t = \tau$(relaxation time) $E = \frac{E_0}{e}$

Substituting the value of E in eq. (2), we get

From eq. (2), $\frac{E_0}{e} = E_0\, e^{-2b\tau}$

$e^{-1} = e^{-2b\tau}$

$-1 = -2\, b\, \tau$

$\therefore\ \boldsymbol{\tau = \left(\frac{1}{2}\, b\right)}$ **---------- (3)**

From eq. (2), $\mathbf{E = E_0 e^{-t/\tau}}$ **------- (4)**

Power dissipation, $\mathbf{P} = \frac{E}{\tau}$

Quality factor (Q):

Quality factor (Q) defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

i.e., $Q = 2\pi \frac{Energy\ stored\ in\ the\ system}{Energy\ lost\ per\ period}$

$Q = 2\pi \frac{E}{PT}$, where P is power dissipatedand T is period time

Where E = energy stored

P = power dissipation

T = Time period

We know that, $P = \frac{E}{\tau}$ where $\boldsymbol{\tau}$ **=** relaxation time

So, $Q = \frac{2\pi\, E}{(E/\tau)T} = \frac{2\pi\, \tau}{T}$ $\because$ $\omega = \frac{2\pi}{T}$ =(angular frequency)

$\mathbf{Q} = \omega\, \tau$

Here, Q α **τ,** i.e., the higher the value of Q, the higher would be the value of relaxation time.

FORCED VIBRATIONAS

The vibrations of a body which vibrates with a frequency other than its natural frequency under the action of an external periodic force are called '***forced vibrations***'

"A body executing forced vibrations is called driven oscillator"

EQUATION OF FORCED VIBRATIONS:

The forces acting on the particle are,

1. The restoring force (f_r) is directly proportional to the displacement (x) but in opposite direction.

   i.e, $f_r \alpha - x$ (or) $f_r = -\mu x$

where μ = proportionality constant (or) force constant or force per unit displacement

2. The frictional force (f_2) proportional to velocity (υ) but in opposite direction

   i.e, $f_f \alpha -\upsilon$ (or) $f_f \alpha - \frac{dx}{dt} \because \frac{dx}{dt} = \upsilon$

   (or) $f_2 = -r\frac{dx}{dt}$, where r = frictional force per unit velocity

3. The external periodic force $f_e = F \text{ Sin } pt$

   where F = maximum value of the force,

   p = 2π n = driving frequency (or) $n = \frac{p}{2\pi}$ = frequency

$\therefore$ The Total force acting on the particle, $f_t = f_r + f_f + f_e$

$$\boldsymbol{f_t = -\mu x - r\frac{dx}{dt} + F \text{ Sin } pt}$$

The impressed periodic force is called driver and the body executing forced vibrations is called Driven Oscillators.

By Newtons's second law of motion, it is equal to the product of mass *m* of the particle and instantaneous acceleration i.e., $m\frac{d^2x}{dt^2}$, hence

But, $f_t = m a$ Where, m = mass of the particle

$$f_t = m\frac{d^2x}{dt^2}$$

$$\therefore \quad m\frac{d^2x}{dt^2} = -\mu x - r\frac{dx}{dt} + F \text{ Sin } pt$$

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \mu x = F \text{ Sin } pt$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{\mu}{m}x = \frac{F}{m}\text{ Sin pt}$$

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\text{Sin pt} \quad \text{--------(1)}$$

where, $\frac{r}{m} = 2b$, $\frac{F}{m} = f$, $\omega^2 = \frac{\mu}{m}$ (or) $\omega = \sqrt{\frac{\mu}{m}}$

This is the differential equation of forced vibrations.

SOLUTION OF EQUATION OF FORCED OSCILLATIONS

(Amplitude and Phase of forced Vibrations)

When a steady state is set up, the particle vibrates with the frequency of applied force, and not with its own natural frequency. The solution of differential eq. (1) is of the type

$$x = A\text{ Sin }(pt - \theta) \quad \text{--------(2)}$$

where A is the steady amplitude of vibration and θ is the angle by which the displacement x lags behind the applied force F sin pt. A and θ are arbitrary constants.

Differentiating eq. (2), we have,

$$\frac{dx}{dt} = A\,p\text{ Cos }(pt - \theta)$$

$$\frac{d^2x}{dt^2} = -A\,p^2\text{ Sin }(pt - \theta)$$

Substituting the values of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in eq (1), we get

$$-A\,p^2\text{ Sin }(pt - \theta) + 2bA\,p\text{ Cos }(pt - \theta) + \omega^2 A\text{ Sin }(pt - \theta) = f\text{ Sin pt} = f\text{Sin }[(pt - \theta) + \theta]$$

$$A(\omega^2 - p^2)\text{ Sin }(pt - \theta) + 2bAp\text{ Cos }(pt - \theta) = f\text{Sin }(pt - \theta)\text{ Cos}\theta + f\text{ Cos }(pt - \theta)\text{ Sin }\theta$$

The relation holds good for all values of t, the coefficients of Sin (pt–θ)' and 'Cos (pt–θ) terms on both sides of the equation are equal i.e.,

Comparing the coefficients of 'Sin (pt–θ)' and 'Cos (pt–θ)' on both sides, we get

$$A(\omega^2 - p^2) = f\text{ Cos}\theta \quad \text{--------(3)}$$

$$2bA\,p = f\text{ Sin }\theta \quad \text{--------(4)}$$

Squaring and adding equations (3) & (4) we get,

$$A^2(\omega^2 - p^2)^2 + 4b^2A^2p^2 = f^2$$

$$A^2[(\omega^2 - p^2)^2 + 4b^2p^2] = f^2$$

Amplitude of forced vibration, A= $\frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$ --------(5)

Dividing equation (4) with (3) we get,

$$\text{Tan}\theta = \frac{2bAp}{A(\omega^2-p^2)} = \frac{2bp}{(\omega^2-p^2)}$$

Phase of vibration , $\theta = Tan^{-1}\left(\frac{2bp}{(\omega^2-p^2)}\right)$ --------(6)

Substituting the value of A from eq (5) in eq. (2)

$\therefore$ x = $\frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$ Sin (pt –θ) --------(7)

Note: p = driving frequency of applied force = 2π n, &ω = $\sqrt{\frac{\mu}{m}}$

Depending upon the relative values of *p* and *ω*, three cases are possible:

Different cases of Amplitude and Phase

<u>Case (1):</u>When driving frequency is low i.e., p << ω. In this amplitude of vibrations are given by

Amplitude, A= $\frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \approx \frac{f}{\omega^2}$ = Constant

and $\theta = Tan^{-1}\left(\frac{2bp}{(\omega^2-p^2)}\right) = \text{Tan}^{-1}(0) \approx 0$

This shows, the amplitude is independent of frequency of force. It depends on magnitude of applied force and force constant 'μ'

The force and displacement are always in phase i.e., in the same phase.

Case (2):**When p =ω,**i.e., frequency of force is equal to the frequency of particle (or) body

In this case, the Amplitude of vibration is,

A= $\frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} = \frac{f}{2bp} = \frac{F}{m\frac{r}{m}\omega} = \frac{F}{r\omega}$ [$\because \frac{r}{m}$ =2 b, $\frac{F}{m}$ = f and p = ω]

$\theta = Tan^{-1}\left(\frac{2bp}{0}\right) = \text{Tan}^{-1}(\infty) = \frac{\pi}{2}$

Thus, the amplitude of vibration is depends on 'damping force' and for small damping forces, the amplitude will be quite large. The displacement lags behind the force by $\frac{\pi}{2}$.

<u>Case (3):</u>When p >ω, i.e., the frequency of force is greater than the natural frequency ω of the body.

In this case, Amplitude, A= $\frac{f}{\sqrt{p^2+4b^2p^2}} \approx \frac{f}{p^2} \approx \frac{F}{mp^2}$ $\because \frac{F}{m} = f$

$$\theta = Tan^{-1}\left(\frac{2bp}{-p^2}\right) = Tan^{-1}\left(\frac{-2b}{p}\right) \approx \mathrm{Tan}^{-1}(-0) = \pi$$

Thus, the amplitude A goes on decreasing and phase difference tends towards 'π'.

Resonance:

The phenomenon of making a body vibrates with its natural frequency under the influence of another vibrating body with the same frequency is called resonance.

Example:

1. Tuning a radio (or) transistor, when natural frequency is so adjusted, by moving the tuning knob of the receiver set that it equals the frequency of the radio waves, the resonance takes place and the incoming sound waves can be listened after being amplified.

2. Musical instrument can be made to vibrate by bringing them in contact with vibrations which have the frequency equal to the natural frequency of the instrument.

3. Soldiers crossing a suspension bridge are prohibited to march in steps and areadvised to march on suspension bridges out of steps so as to avoid the resonance between the natural frequency of the bridge and the frequency of steps of soldiers which may cause the collapse of the bridge.

AMPLITUDE RESONANCE:

The amplitude of forced oscillations varies with the frequency of applied force and becomes maximum at a particular frequency, this phenomenon is called amplitude resonance.

Conditions of Amplitude Resonance

In case of forced vibrations,

Amplitude, $A = \dfrac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2 p^2}}$ ----------------- **(1)**

and $\theta = Tan^{-1}\left(\dfrac{2bp}{(\omega^2 - p^2)}\right)$ ----------------- **(2)**

Eq.(1) shows that the amplitude varies with the frequency of force (p).

For particular value of 'p' amplitude becomes maximum, this is called amplitude resonance.

The amplitude is maximum when the term $\sqrt{(\omega^2 - p^2)^2 + 4b^2 p^2}$ becomes minimum.

(or) $\dfrac{d}{dp}\left[(\omega^2 - p^2)^2 + 4b^2 p^2\right] = 0$

$$2(\omega^2 - p^2)(-2p) + 4b^2(2p) = 0$$

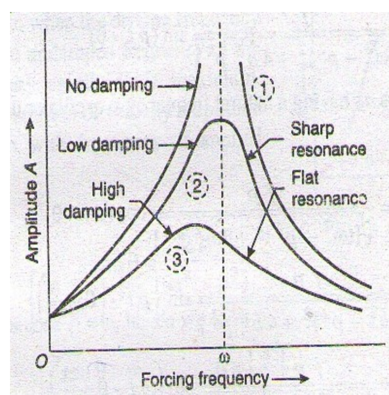
$$(\omega^2 - p^2) = 2b^2$$

$p^2 = \omega^2 - 2b^2$ (or) $p = \sqrt{(\omega^2 - 2b^2)}$ -------------**(3)**

Thus, the amplitude is maximum when frequency $\left(\dfrac{p}{2\pi}\right)$ of the impressed force becomes $\dfrac{\sqrt{(\omega^2 - 2b^2)}}{2\pi}$.

This is the resonant frequency.

It gives frequency of the system both in presence of damping i.e., $\dfrac{\sqrt{(\omega^2 - 2b^2)}}{2\pi}$ and in the absence of damping i.e., $\frac{\omega}{2\pi}$. If the damping is small, then it is neglected and the condition of maximum amplitude is reduced to $p = \omega$.



Substituting the condition (3) in eq. (1), we get

$$A_{max} = \frac{f}{\sqrt{(\omega^2 - \omega^2 + 2b^2)^2 + 4b^2(\omega^2 - 2b^2)}}$$

$$A_{max} = \frac{f}{\sqrt{4b^4 + 4b^2\omega^2 - 8b^4}} = \frac{f}{\sqrt{4b^2\omega^2 - 4b^4}}$$

$$A_{max} = \frac{f}{2b\sqrt{\omega^2 - b^2}} = \frac{f}{2b\sqrt{p^2 + 2b^2 - b^2}}$$

$\because p^2 = \omega^2 - 2b^2$

(or) $\omega^2 = p^2 + 2b^2$

$$A_{max} = \frac{f}{2b\sqrt{p^2 + b^2}}$$

For low damping, $A_{max} \approx \frac{f}{2bp}$

Then, $A_{max} \rightarrow \infty$ as $b \rightarrow 0$

In figure, curve (1) shows amplitude when there is no damping i.e., b = 0. The amplitude becomes infinite at $p=\omega$. It can never be attained in practise due to frictional resistance as slight damping is always present.

Curves (2) & (3) shows the effect of damping on the amplitude. It is observed that peak of the curve moves towards the left and the value of A, which is different for different values of b (damping), diminishes as the value of b increases.

For smaller values of b, the fall in the curve about $p=\omega$ is steeper than for large values, i.e., smaller is the value of damping, greater is the departure of amplitude of forced vibrations from the maximum value vice-versa.

Problems:

1. The differential equation for a certain system is $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$ if $\omega >> k$, find the time in which amplitude falls to 1/e times the initial value?

Sol: The given equation is of damped harmonic motion.

The amplitude is given by, $a = a_0 e^{-bt} = a_0 e^{-kt}$

According to given problem, $a = \frac{a_0}{e}$

$\frac{a_0}{e} = a_0 e^{-kt}$ (or) $e^{-1} = e^{-kt}$

$$k t = 1 \quad \text{(or)} \quad t = \frac{1}{k} \text{ sec}$$

2. The damped oscillator starting from rest reaches first amplitude of 500mm. It reduces to 50mm after 100 oscillations. The periodic time is 2.3 sec. Find the damping constant and relaxation time?

Sol: Given that, T = 2.3 sec

The amplitude is given by, $a = a_0 e^{-bt}$

The first amplitude, $a_1 = a_0 e^{-bT/4}$ (for 1^{st} amplitude, t=T/4)

The 201^{th} amplitude, $a_{201} = a_0 e^{-b(100T + T/4)}$ (for 201^{th} amplitude, t =100T + T/4)

(After 100 oscillations 201^{th} amplitude is obtained)

$a_1 = 500$ mm and $a_{201} = 50$ mm

$\frac{a_{201}}{a_1} = e^{-100\,bT}$

$\frac{50}{500} = e^{-100\,bT}$ (or) $e^{100\,bT} = 10$

$$100\,bT = \log_e 10 = 2.303 \log_{10} 10 = 2.303$$

$$100\,b \times 2.3 = 2.303$$

$$\text{Damping constant, } b \approx \frac{1}{100} = 10^{-2} \text{ sec.}$$

$$\text{Relaxation time, } \tau = \frac{1}{2b} = \frac{1}{2x10^{-2}} = \frac{100}{2} = 50 \text{ sec.}$$

3. The quality factor of a sonometer wire is 2×10^3. On plucking it makes 240 vibrations per second. Calculate the time in which amplitude decreases to half the initial value?

Sol: Given that, $Q = 2 \times 10^3$ and $\nu = 240$ Hz

The quality factor, $Q = \omega \tau$

$$= 2\pi\nu\tau = 2 \times 3.14 \times 240\,\tau$$

$$2 \times 10^3 = 2 \times 3.14 \times 240\,\tau$$

$$\tau = \frac{2\times10^3}{2\times3.14\times240} = 1.327 \text{ sec}$$

But, $\tau = \frac{1}{2b}$ (or) $\frac{1}{b} = 2\tau = 2 \times 1.327 = 2.654$

The amplitude of damped vibrations is, $a = a_0\, e^{-bt}$

$\frac{a}{a_0} = e^{-bt}$ given that, $a = \frac{a_0}{2}$

$\frac{1}{2} = e^{-bt}$ (or) $e^{bt} = 2$ (or) $b\,t = \log_e 2 = 2.303\log_{10} 2$

$$b\,t = 2.303 \times 0.3010 = 0.6932$$

$$t = \frac{0.6932}{b} = 0.6932 \times 2.654 = 1.84 \text{ sec.}$$

4. The amplitude of a seconds pendulum falls to half of its initial value in 150 seconds. Calculate quality factor?

Sol: The amplitude of damped vibrations is, $a = a_0\, e^{-bt}$

$\frac{a}{a_0} = e^{-bt}$ given that, $a = \frac{a_0}{2}$ and $t = 150$ sec

$\frac{1}{2} = e^{-150\,t}$ (or) $e^{150\,t} = 2$ (or) $150\,b = \log_e 2 = 2.303\log_{10} 2$

$$150\,b = 2.303 \times 0.3010 = 0.6932$$

$$b = \frac{0.6932}{150} = 0.00462$$

For seconds pendulum, T = 2 sec

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi = 3.14$$

$$\tau = \frac{1}{2b} = \frac{1}{0.00924}$$

So, the quality factor, $Q = \tau\,\omega = \frac{3.14}{0.00924} \approx 340$

5. The quality factor of an oscillator is 500.Find its initial energy of its amplitude 0.01 m. Also calculate the energy lost in first cycle? Given that $s = m\,\omega^2 = 100$ N/m

Sol: Given that $s = m\,\omega^2 = 100$ N/m

$Q = 500$

Amplitude, $a = 0.01$ m

The initial energy of an oscillator, $E = \frac{1}{2} m\,\omega^2 a^2 = \frac{1}{2} s\,a^2$

$$E = \frac{1}{2} \times 100 \times (0.01)^2$$

$$E = 5 \times 10^{-3}\ \text{J}$$

The quality factor, $Q = \dfrac{2\pi\ \textit{energy stored in system}}{\textit{energy lost per period}}$

$$500 = \frac{2\pi E}{\textit{energy lost per period}}$$

Energy lost in first cycle (or) per period $= \dfrac{2\pi E}{500}$

$$= \frac{2 \times 3.14 \times 5 \times 10^{-3}}{500}$$

$$= 6.28 \times 10^{-5}\ \text{J}$$

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UNIT-III
COMPLEX VIBRATIONS

FOURIER'S THEOREM:

Any single valued, finite, continuous periodic function can be represented as a summation of an infinite number of simple harmonic terms having frequencies which are multiples of the frequency of the function.

Mathematically,

$y = f(\omega t) = A_0 + A_1 \text{ Cos } \omega t + A_2 \text{ Cos } 2\omega t + A_3 \text{ Cos } 3\omega t + \ldots + A_r \text{ Cos } r\omega t + \ldots\ldots + B_1 \text{ Sin } \omega t + B_2 \text{ Sin } 2\omega t + B_3 \text{ Sin } 3\omega t + \ldots + B_r \text{ Sin } r\omega t + \ldots$

$$y = f(\omega t) = A_0 + \sum_{r=1}^{\infty} \left(A_r Cos\ r\omega\ t + B_r Sin\ r\omega\ t \right) \quad \text{-------- (1)}$$

Where $y = f(t)$ = the displacement of a complex periodic motion of angular frequency 'ω'

$A_1, A_2, A_3, .. A_r \ldots\ldots B_1, B_2, B_3 \ldots\ldots B_r \ldots$ are constants

A_0 = The constant representing the displacement of the axis of motion from the time axis.

Evaluation of A_0:

In order to evaluate $\mathbf{A_0}$, multiply eq. (1) with 'dt' and integrate between the limits t = 0 and t = T, where, T = period of the function. Hence,

$$\int_0^T f(\omega\ t)\ dt = A_0 \int_0^T dt + A_1 \int_0^T Cos\,\omega t\ dt + \ldots\ldots + A_r \int_0^T Cos\ r\omega t\ dt + B_1 \int_0^T Sin\,\omega t\ dt + \ldots\ldots + B_r \int_0^T Sin\ r\omega t\ dt$$

$$\int_0^T f(\omega\ t)\ dt = A_0 T, \text{ all other integrals being zero}$$

$$A_0 = \frac{1}{T} \int_0^T f(\omega t)\ dt \quad \text{-------- (2)}$$

Evaluation of A_r:

In order to evaluate $\mathbf{A_r}$, multiply eq. (1) with 'Cos rωt dt' and integrate between the limits t = 0 to t = T, we get,

$$\int_0^T f(\omega t)\ Cos r\ \omega t\ dt = A_0\int_0^T Cos r\omega t\ dt + A_1\int_0^T Cos\,\omega t\ Cos r\omega t\ dt + \ldots + A_r\int_0^T Cos^2 r\omega t\ dt +$$

$$B_1\int_0^T Sin\ \omega t . Cos\omega t dt + \ldots\ldots + B_r\int_0^T Sin\ r\omega t\ Cos r\omega t\ dt$$

$$= A_r\int_0^T Cos^2 r\omega t\ dt\ , \quad \text{all other integrals being zero}$$

$$\int_0^T f(\omega t)\ Cos r\ \omega t\ dt = A_r\int_{t=0}^T \left(\frac{1+Cos 2r\omega t}{2}\right) dt \because Cos^2\theta = \frac{1+Cos 2\theta}{2}$$

$$= \frac{A_r}{2}\left[t + \frac{Sin\,2r\omega t}{2r\omega}\right]_{t=0}^T = \frac{A_r T}{2} \because\ Sin\ 2\pi r = 0,\ \omega = \frac{2\pi}{T}$$

$$\therefore\ \mathbf{A_r} = \frac{2}{T}\int_0^T f(\omega t)\ Cos r\omega t\ dt \text{ -------- } \mathbf{(3)}$$

Evaluation of B_r:

In order to evaluate B_r, multiply eq. (1) with 'Sin rωt dt' and integrate between the limits t = 0 and t = T where, T = period of the function

$$\int_0^T f(\omega t)\ Sin r\,\omega t\ dt = A_0\int_0^T Sin r\,\omega t\ dt + A_1\int_0^T Cos\,\omega t\ Sin r\,\omega t\ dt + \ldots\ldots +$$

$$A_r\int_0^T Cos\ \ r\omega t\ Sin r\,\omega t\ dt + B_1\int_0^T Sin\,\omega t\ Sin r\omega t\ dt + \ldots + B_r\int_0^T Sin^2 r\omega t\ dt$$

$$= B_r\int_0^T Sin^2 r\omega t\ dt\ , \quad \text{all other integrals being zero}$$

$$\int_0^T f(\omega t)\ Sin r\,\omega t\ dt = B_r\int_{t=0}^T \left(\frac{1-Cos 2r\omega t}{2}\right) dt \because Sin^2\ \theta\ = \frac{1-Cos 2\theta}{2}\int_0^T f(\omega t)\ Sin r\,\omega t\ dt =$$

$$\frac{B_r}{2}\left[t - \frac{Sin\,2r\omega t}{2r\omega}\right]_{t=0}^T = \frac{B_r T}{2} \because\quad Sin\ 4\pi r = 0,\quad \omega = \frac{2\pi}{T}$$

$$\therefore\ \mathbf{B_r} = \frac{2}{T}\int_0^T f(\omega t)\ Sin r\omega t\ dt \text{ -------- } \mathbf{(4)}$$

[Note: $\int_0^T Cos\ \ r\omega t\ dt = \left[\frac{Sin r\ \omega t}{r\omega}\right]_{t=0}^T = \frac{1}{r\omega}\left(Sin 0 - Sin 2\pi\ r\right) = 0$

$\because$ Sin 2πr = 0, ω = $\frac{2\pi}{T}$

$$\int_0^T Sin\ r\omega t\ dt = \left[\frac{Cosr\ \omega t}{r\omega}\right]_{t=0}^T = \frac{1}{r\omega}\left(Cos\,0 - Cos\,2\pi\ r\right) = \frac{1}{r\omega}(1-1) = 0$$

$\because$ Cos 2πr = 1, ω = $\frac{2\pi}{T}$

$$\int_0^T Sin\ r\omega t\ Cosr\omega t\ dt = \frac{1}{2}\int_{t=0}^T Sin2r\omega t\ dt \quad \because \text{Sin } \theta \text{ Cos } \theta = \frac{1}{2} \text{ Sin } 2\theta$$

$$= -\frac{1}{2}\left[\frac{Cos2r\omega t}{2r\omega}\right]_{t=0}^T \quad \because \text{ Cos } 4\pi r = 1,\ \omega = \frac{2\pi}{T}$$

$$= -\frac{1}{4r\omega}\left(Cos\,0 - Cos\,4\pi\ r\right) = -\frac{1}{4r\omega}(1-1) = 0$$

1. Sin A Sin B = $\frac{1}{2}$[Cos (A–B) – Cos (A+B)]

2. Cos A Cos B = $\frac{1}{2}$[Cos (A-B) + Cos (A+B)]

3. Sin A Cos B = $\frac{1}{2}$[Sin (A+B) + Sin (A–B)]

4. Cos A Sin B = $\frac{1}{2}$[Sin (A+B) – Sin (A–B)]

Note: $\int_0^T Sin\omega t\ Sinr\omega t\ dt = \frac{1}{2}\int_{t=0}^T [Cos(\omega t - r\omega t) - Cos(\omega t + r\omega t)]dt$

$$= \frac{1}{2}\int_{t=0}^T Cos(1-r)\omega t dt - \frac{1}{2}\int_{t=0}^T Cos(1+r)\omega t dt = 0$$

$$\int_0^T Cos\,\omega t\ Cosr\,\omega t\ dt = \frac{1}{2}\int_{t=0}^T [Cos(\omega t - r\omega t) + Cos(\omega t + r\omega t)]dt$$

$$= \frac{1}{2}\int_{t=0}^T Cos(1-r)\omega t dt + \frac{1}{2}\int_{t=0}^T Cos(1+r)\omega t dt = 0$$

$$\int_0^T Sin\,\omega t\ Cosr\,\omega t\ dt = \frac{1}{2}\int_{t=0}^T [Sin(\omega t + r\omega t) + Sin(\omega t - r\omega t)]dt$$

$$= \frac{1}{2}\int_{t=0}^T Sin(1+r)\omega t dt + \frac{1}{2}\int_{t=0}^T Sin(1-r)\omega t dt = 0]$$

Limitations of Fourier's theorem:

(i) **The function should be finite**

i.e., the displacement should always have finite values and should never be infinite at any time.

(ii) **The function should be single-valued** i.e., the displacement should have only one value at a given instant 't'

(iii) **The function should be continuous**

i.e., the function should have a finite number of jumps within its time- interval

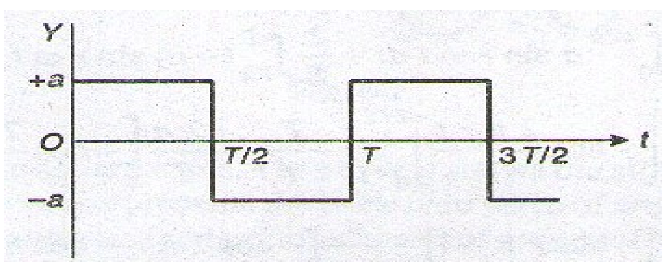
Fourier series of a function f (ω t) between the limits – π to +π, is

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\omega t)\,dt$$

$$A_r = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\omega t)\,Cos\,r\,\omega t\,dt$$

$$B_r = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\omega t)\,Sin\,r\,\omega t\,dt$$

FOURIER ANALYSIS OF SQUARE WAVE:



A square wave is shown in figure, the displacement is along the Y-axis and the time is along X-axis. The function has a constant value 'a' from t =0 to t $=\frac{T}{2}$ and '–a' from t $=\frac{T}{2}$ to t = T.

So, y = f (ω t) = a when t =0 to t = t $=\frac{T}{2}$

And y = f (ω t) = – a when t =t $=\frac{T}{2}$ to t = T

Calculation of values A_0, A_r and B_r

The Value of A_0:

Here, the axis of vibration coincides with the time axis and hence $A_0 = 0$

$$A_0 = \frac{1}{T}\int_0^T f(\omega t)\,dt = \frac{1}{T}\int_0^{T/2} f(\omega t)\,dt + \frac{1}{T}\int_{T/2}^{T} f(\omega t)\,dt$$

$$= \frac{1}{T}\int_0^{T/2} a \; dt + \frac{1}{T}\int_{T/2}^{T} (-a) \; dt$$

$$= \frac{a}{T}\left(t\right)_{t=0}^{T/2} - \frac{a}{T}\left(t\right)_{t=T/2}^{T}$$

$$A_0 = \frac{a}{2} - a + \frac{a}{2} = a - a = 0$$

The Value of A_r:

$$A_r = \frac{2}{T}\int_0^{T} f(\omega t) \; Cos r\omega t \; dt$$

$$A_r = \frac{2}{T}\int_0^{T/2} a \; Cos r\omega t dt + \frac{2}{T}\int_{T/2}^{T} (-a) \; Cos r\omega t dt$$

$$A_r = \frac{2\,a}{T}\int_0^{T/2} Cos\left(\frac{2\pi r}{T}\right) t dt - \frac{2\,a}{T}\int_{T/2}^{T} Cos\left(\frac{2\pi r}{T}\right) t dt \quad \because \omega = \frac{2\pi}{T}$$

$$A_r = \frac{2\,a}{T}\left[Sin\left(\frac{2\pi\; rt}{T}\right)\right]_{t=0}^{T/2} \frac{T}{2\pi\; r} - \frac{2\,a}{T}\left[Sin\left(\frac{2\pi\; rt}{T}\right)\right]_{t=T/2}^{T} \frac{T}{2\pi\; r}$$

$$A_r = \frac{a}{\pi r}[\text{Sin } r\pi - 0] - \frac{a}{\pi r}[\text{ Sin } 2\pi \text{ r} - \text{Sin } r\pi]$$

$$A_r = \frac{a}{\pi r}[\text{Sin } r\pi - \text{Sin } 2\pi \text{ r} + \text{Sin } r\pi]$$

$$A_r = \frac{a}{\pi r}[2 \text{ Sin } r\pi - \text{Sin } 2\pi \text{ r}] = 0 \qquad \because \text{ Sin } 2\pi r = 0$$

The Value of B_r:

$$B_r = \frac{2}{T}\int_0^{T} f(\omega t) \; Sin r\omega t \; dt$$

$$B_r = \frac{2}{T}\int_0^{T/2} a \; Sin r\omega t dt + \frac{2}{T}\int_{T/2}^{T} (-a) \; Sin r\omega t dt$$

$$B_r = \frac{2\,a}{T}\int_0^{T/2} Sin\left(\frac{2\pi r}{T}\right) t dt - \frac{2\,a}{T}\int_{T/2}^{T} Sin\left(\frac{2\pi r}{T}\right) t dt \qquad \because \omega = \frac{2\pi}{T}$$

$$B_r = \frac{2\,a}{T}\left[-Cos\left(\frac{2\pi\; rt}{T}\right)\right]_{t=0}^{T/2} \frac{T}{2\pi\; r} - \frac{2\,a}{T}\left[-Cos\left(\frac{2\pi\; rt}{T}\right)\right]_{t=T/2}^{T} \frac{T}{2\pi\; r}$$

$$B_r = \frac{a}{\pi r}[-\text{ Cos } r\pi + 1] - \frac{a}{\pi r}[-\text{ Cos } 2\pi \text{ r} + \text{Cos } r\pi]$$

$$B_r = \frac{a}{\pi r}[-\text{ Cos } r\pi + 1 + 1 - \text{Cos } r\pi] \quad \because \text{ Cos } 2\pi r = 1$$

$$B_r = \frac{a}{\pi r}[2 - 2\text{Cos } r\pi] = \frac{2a}{\pi r}[1 - \text{Cos } r\pi]$$

When 'r' is even, i.e., r = 2,4, 6…, Cos rπ = 1

$$B_r = \frac{2a}{\pi r}[1 - 1] = 0$$

When 'r' is odd, i.e., r = 1,3, 5…, Cos rπ = –1

$$B_r = \frac{2a}{\pi r}[1 - (-1)] = \frac{4a}{\pi r}$$

$$\therefore B_1 = \frac{4a}{\pi},\ B_3 = \frac{4a}{3\pi},\ B_5 = \frac{4a}{5\pi} \ldots\ldots \text{ and } B_2 = B_4 = B_6 = \ldots\ldots = 0$$

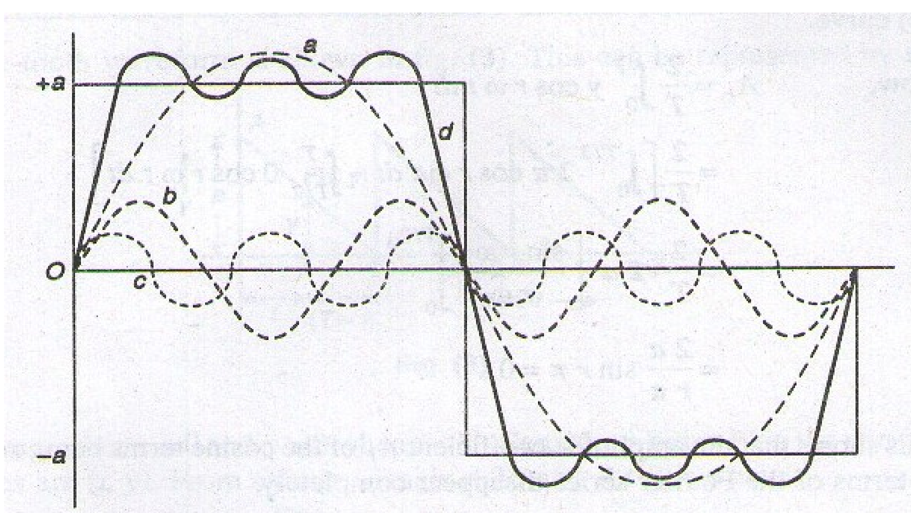
$$y = f(\omega t) = \frac{4a}{\pi}\text{Sin } \omega t + \frac{4a}{3\pi}\text{Sin } 3\omega t + \frac{4a}{5\pi}\text{Sin } 5\omega t + \ldots\ldots$$

$$y = f(\omega t) = \frac{4a}{\pi}[\text{Sin } \omega t + \frac{1}{3}\text{Sin } 3\omega t + \frac{1}{5}\text{Sin } 5\omega t + \ldots\ldots]$$

Component vibrations are shown in figure.

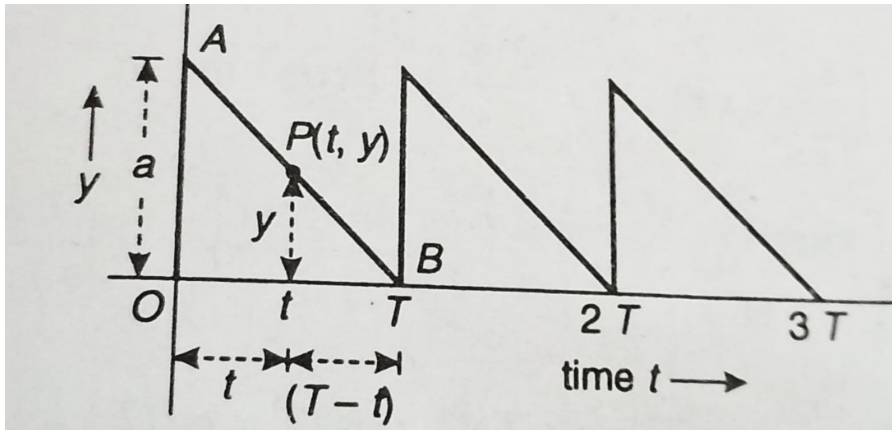
The curve 'a' shows the Simple harmonic wave of angular frequency 'ω'
The curve 'b' shows the Simple harmonic wave of angular frequency '3ω'
The curve 'c' shows the Simple harmonic wave of angular frequency '5ω'

The addition of these curves yields a curve 'd'. Approximately this represents a square wave, we get a better approximation to the original curve if we add more and more terms.



FOURIER ANALYSIS OF SAW-TOOTH WAVE FORM

A Saw-tooth Waveform is represented by a linear relation y= a when t =0 and y= 0 when t =T.



Consider a point P on the curve whose coordinates are (t, y).

From similar triangle AOB and P t B, we have $\frac{a}{y} = \frac{T}{(T-t)}$

or $y = \frac{a(T-t)}{T} = a\left(1-\frac{t}{T}\right) = f(t)$

So, in case of saw-tooth waveform, the displacement at an instant *t* is represented by

$$y = f(t) = a\left(1-\frac{t}{T}\right) \quad \text{for } 0 < t < T \text{ -------------(1)}$$

According to Fourier series,

$$y = f(\omega t) = A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + \text{---------} + A_r \cos r\omega t + \ldots\ldots + B_1 \sin \omega t + B_2 \sin 2\omega t + \text{---------} + B_r \sin r\omega t + \ldots \text{------------(2)}$$

Where, $A_0 = \frac{1}{T}\int_0^T f(t)dt$

$A_r = \frac{2}{T}\int_0^T f(t) \cos r\omega t \, dt$

and $B_r = \frac{2}{T}\int_0^T f(t) \sin r\omega t \, dt$

To calculate the values of the coefficients **A_0, A_r and B_r**

$$A_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^T a\left(1-\frac{t}{T}\right)dt = \frac{a}{T}\left[t - \frac{t^2}{2T}\right]_0^T = \frac{a}{T}\left(T - \frac{T^2}{2T}\right)$$

$$A_0 = \frac{a}{T}\frac{T}{2} = \frac{a}{T} \text{ ------------------(3)}$$

Thus, the axis of the curve is at a distance $\left(\frac{a}{2}\right)$ from the time axis.

For A_r, we have

$$A_r = \frac{2}{T}\int_0^T f(t)\cos r\omega t\, dt \;=\; \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\cos r\omega t\, dt$$

$$A_r = \frac{2a}{T}\int_0^T \cos r\omega t\, dt - \frac{2a}{T^2}\int_0^T t\cos r\omega t\, dt$$

$$A_r = \frac{2a}{T^2}\left[\frac{sinr\omega t}{r\omega}\right]_0^T - \frac{2a}{T^2}\left[\left\{r\left(\frac{sinr\omega t}{r\omega}\right)_0^T - \int_0^T \frac{sinr\omega t}{r\omega}\, \mathrm{dt}\right\}\right]$$

$$A_r = 0 - \frac{2a}{T^2}\left[\mathrm{t}\,\frac{\sin\left(\frac{2\pi rt}{T}\right)}{2\pi r/t} + \frac{\cos\left(\frac{2\pi rt}{T}\right)}{2\pi r/T}\right]_0^T$$

Since, $[\sin r\,\omega t]_0^t = 0$ where $\omega = \frac{2\pi}{T}$

$$A_r = \frac{2a}{T^2}\left[\mathrm{T}\,\frac{\sin 2\pi r}{2\pi r/T} - 0 + \frac{cos 2\pi}{2\pi\pi r/T^2} - \frac{\cos 0}{2\pi r/T^2}\right]$$

$$A_r = \frac{2a}{T^2}\left[\frac{1}{\left(\frac{2\pi r}{T}\right)^2} - \frac{1}{\left(\frac{2\pi r}{T}\right)^2}\right]$$

Since, $\sin 2\pi r = 0$ and $\cos 2\pi r = 1$

$A_r = 0$

Hence, all cosine terms of Fourier series have zero amplitude.

For B_r, we have,

$$B_r = \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\sin r\omega t\, dt = \frac{2a}{T}\int_0^T sin\left[\frac{2\pi rt}{T}\right]dt - \frac{2a}{T^2}\int_0^T sin\left[\frac{2\pi rt}{T}\right]dt$$

$$B_r = \frac{2a}{T^2}\int_0^T t sin\left[\frac{2\pi rt}{T}\right]dt \qquad \text{Since, } \int_0^T sin\left[\frac{2\pi rt}{T}\right]dt = 0$$

Integrating by parts,

$$B_r = \frac{2a}{T^2}\left[\left\{\mathrm{t}\,\frac{-\cos 2\pi rt/T}{2\pi r/T}\right\}_0^T - \int_0^T \frac{-\cos 2\pi rt/T}{2\pi r/T}\, dt\right]$$

$$B_r = \frac{2a}{T^2}\left[\mathrm{t}\frac{\cos 2\pi rt/T}{2\pi r/T} - \frac{\sin 2\pi rt/T}{(2\pi r/T)^2}\right]_0^T$$

$$B_r = \frac{2a}{T^2}\left[\mathrm{T}\frac{\cos 2\pi r}{2\pi r/T} - 0 - \frac{\sin 2}{(2\pi r/T)^2} + \frac{\sin 0}{(2\pi r/T)^2}\right]$$

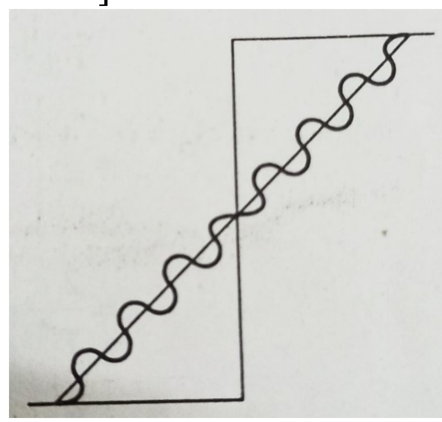
$B_r = \frac{2a}{T^2}\left[T\frac{1}{(\frac{2\pi r}{T})}\right] = \frac{a}{r\pi}$[since, $\cos 2\pi r = 1$ *and* $\sin 2\pi r = 0$]

$\therefore \mathbf{B}_1 = \left(\frac{a}{\pi}\right)$, $\mathbf{B}_2 = \left(\frac{a}{2\pi}\right)$, $\mathbf{B}_3 = \left(\frac{a}{3\pi}\right)$ and so on.

Hence, the complete vibration is represented by,

$y = f(t) = \frac{a}{2} + \frac{a}{\pi}\sin\omega t + \frac{a}{2\pi}\sin 2\omega t + \frac{a}{3\pi}\sin 3\omega t$ +---------

$y = f(t) = \frac{a}{2} + \frac{a}{\pi}\left[\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + - - - - - -\right]$



having frequencies in the ratio 1:2:3…… and amplitudes in the ratio $1 : \frac{1}{2} : \frac{1}{3}$ and so on.

The addition of successive terms of the series in indicated in the figure. It is observed that if a greater number of terms are used then there is close resemblance between the resultant curve and the curve under analysis.

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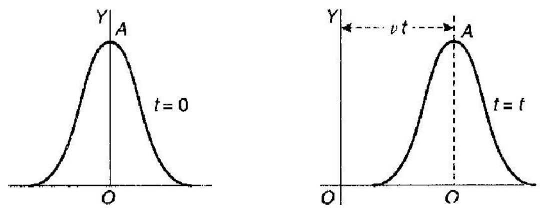
UNIT-IV

IV (A) VIBRATING STRINGS

GENERAL WAVE EQUATION AND ITS SOLUTION:



(a) Pulse at time $t = 0$ (b) Pulse after $t = t$

Let the pulse is travelling to the right with a velocity 'ʋ'. After a time 't' the pulse reaches a distance 'ʋt' along X- axis.

The wave is be represented as, $y = f(x - \upsilon t)$

The variable y depends on *x* and *t* and hence it is written as, $y(x,t)$

$\therefore y(x, t) = f(x - \upsilon t)$ (from Galilean transformations)

Hence, $y(x, t) = f(x - \upsilon t)$ wave travelling in positive X- axis

$y(x, t) = f(x + \upsilon t)$ wave travelling in negative X- axis

$\therefore y = f(x \pm \upsilon t)$ ------- (1)

Now, we consider the special case, the variable is a harmonic function,

$\therefore y(x, t) = A_o \,\text{Sin}\, [k(x - \upsilon t)]$

Let, 'x' is replaced by $\left(x + \frac{2\pi}{k}\right)$, then

$$y(x, t) = A_o \,\text{Sin}\, [k(x + \frac{2\pi}{k} - \upsilon t)]$$

$$= A_o \,\text{Sin}\, [k(x - \upsilon t) + 2\pi]$$

$$y(x, t) = A_o \,\text{Sin}\, [k(x - \upsilon t)] \quad (\because \text{Sin}(2\pi + \theta) = \text{Sin}\,\theta)$$

The replacement of 'x' by $\left(x + \frac{2\pi}{k}\right)$ gives same value of 'y'

In other words, $\lambda = \frac{2\pi}{k}$

or, $k = \frac{2\pi}{\lambda}$ where, k = wave number

From eq. (1), we consider that, $y = f(\upsilon t \pm x)$ ----------- (2)

Partial differentiating eq.(2) w.r.to 'x' twice, then

$$\frac{\partial y}{\partial x} = \pm f^1(\upsilon\, t \pm x)$$

$$\frac{\partial^2 y}{\partial x^2} = \pm f^{11}(\upsilon\, t \pm x) \text{ ----------- (3)}$$

Where, f^1 and f^{11} are some functions of (ʋt ± x)

Now, again Partial differentiating eq.(2) w.r.to 't' twice, then

$$\frac{\partial y}{\partial t} = \upsilon\, f^1(\upsilon\, t \pm x)$$

$$\frac{\partial^2 y}{\partial t^2} = \upsilon^2 f^{11}(\upsilon\, t \pm x) \text{ ----------- (4)}$$

Fromequations (3) & (4) we get, $\frac{\partial^2 y}{\partial t^2} = \upsilon^2 \frac{\partial^2 y}{\partial x^2}$

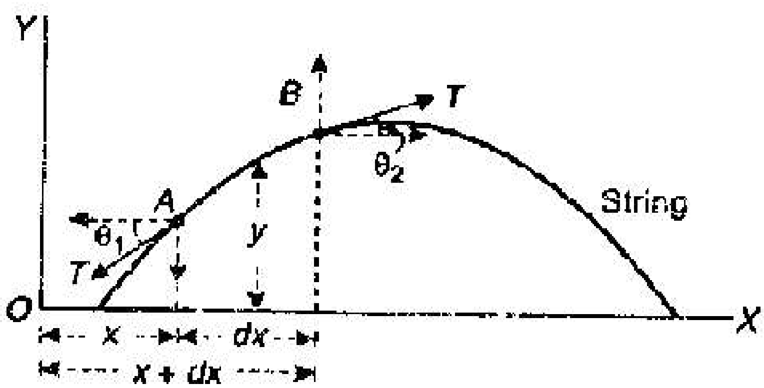
This is called the differential form of the wave equation

General Solution of The Wave Equation:

The arbitrary function either (ʋt - x) or (ʋt + x) will be the solution of the wave equation

$$y = f_1(\upsilon\, t - x) + f_2(\upsilon\, t + x)$$

Velocity of Transverse Wave Along A Stretched String:



A string is fixed between two rigid supports and stretched under a tension 'T' along X- axis. In displaced position, consider a infinitesimal string element AB of length 'dx' between the coordinates x and x+dx as shown

Let 'y' be its displacement at time 't'

Let θ_1 and θ_2 be the angles which the tension (T) makes with X- axis

The components of 'T' in horizontal and vertical directions at A are T Cos θ_1 and T Sin θ_1 and at B are T Cos θ_2 and T Sin θ_2 respectively. T Cos θ_1 and T Cos θ_2 are nearly equal and balances each other,

The resultant upward force F in upward direction,

$$F_y = T\sin\theta_2 - T\sin\theta_1$$

$$F_y = T[\sin\theta_2 - \sin\theta_1] \text{ -----------(1)}$$

As 'AB' is small θ_1 and θ_2 are also small,

Hence, $\quad \sin\theta_1 \approx \tan\theta_1 \approx \left(\frac{\partial y}{\partial x}\right)_x$

and $\quad \sin\theta_2 \approx \tan\theta_2 \approx \left(\frac{\partial y}{\partial x}\right)_{x+dx}$

$$\therefore F_y = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x\right] \text{ ----------(2)}$$

Using Taylor's series, we expand $\left(\frac{\partial y}{\partial x}\right)_{x+dx}$, i.e.,

$$\left(\frac{\partial y}{\partial x}\right)_{x+dx} = \left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)dx + \left(\frac{\partial^3 y}{\partial x^3}\right)\frac{(dx)^2}{2!} + \ldots \text{ ------------(3)}$$

Neglecting high power terms, we have,

Substituting the values of $\left(\frac{\partial y}{\partial x}\right)_{x+dx}$ from equations (2) & (3) we get,

$$F_y = T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)dx - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$\therefore F_y = T\left(\frac{\partial^2 y}{\partial x^2}\right)dx \text{ ---------(4)}$$

Let m = mass per unit length of the wire

Mass of the element 'AB' = m dx

Force, F_y = mass X acceleration (a)

$$F_y = (m\,dx)\left(\frac{\partial^2 y}{\partial t^2}\right) \text{ ---------(5)} \because a = \left(\frac{\partial^2 y}{\partial t^2}\right)$$

From equations (4) &(5) we get, $m\left(\frac{\partial^2 y}{\partial t^2}\right)dx = T\left(\frac{\partial^2 y}{\partial x^2}\right)dx$

$$\therefore \left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{T}{m}\left(\frac{\partial^2 y}{\partial x^2}\right) \text{---------(6)}$$

The differential equation of a wave motion is

$$\left(\frac{\partial^2 y}{\partial t^2}\right) = \upsilon^2 \left(\frac{\partial^2 y}{\partial x^2}\right) \text{---------(7)}$$

Comparing equations (6) & (7) we get, $\upsilon^2 = \frac{T}{m}$

$$\therefore \upsilon = \sqrt{\frac{T}{m}}$$

This is the velocity of the transverse wave along the string.

MODES OF VIBRATION OF STRETCHED STRING CLAMPED AT BOTH THE ENDS:

Consider a uniform string of length 'ℓ' having mass per unit length 'm' and stretched by a tension 'T'.

The general solution of the wave equation is,

$y = a_1$ Sin $(\omega t - kx) + a_2$ Sin $(\omega t + kx) + b_1$ Cos $(\omega t - kx) + b_2$ Cos $(\omega t + kx)$ ----(1)

where, a_1, a_2, b_1 and b_2 are arbitrary constants.

As the string is fixed at both ends, the boundary conditions are,

$y = 0$ at $x = 0$ for any time 't' -------(2)

$y = 0$ at $x = \ell$ for any time 't' -------(3)

Applying boundary conditions from eqs (1) & (2) we get,

$0 = a_1$ Sin $\omega t + a_2$ Sin $\omega t + b_1$ Cos $\omega t + b_2$ Cos ωt

$0 = (a_1 + a_2)$ Sin $\omega t + (b_1 + b_2)$ Cos ωt

As, Sin $\omega t \neq 0$ and Cos $\omega t \neq 0$

$a_1 + a_2 = 0$ and $b_1 + b_2 = 0$

Thus, we have $a_1 = -a_2$ and $b_1 = -b_2$

Now Eq. (1) becomes

$y = a_1$[Sin $(\omega t - kx)$ –Sin $(\omega t + kx]$ + b_1[Cos $(\omega t - kx)$ – Cos $(\omega t + kx)$]

$y = a_1$ [(Sin ωt Cos kx – Cos ωt Sin kx) – (Sin ωt Cos kx + Cos ωt Sin kx)]+

b_1 [(Cos ωt Cos kx + Sin ωt Sin kx) – (Cos ωt Cos kx – Sin ωt Sin kx)]

$y = -2a_1 \text{Cos } \omega t \text{ Sin } kx + 2b_1 \text{ Sin } \omega t \text{ Sin } kx$

$y = (-2a_1 \text{Cos } \omega t + 2b_1 \text{ Sin } \omega t) \text{ Sin } kx$ -----------(4)

The solution now consists of two terms, i.e., on t and x. Thus, the first boundary condition reduces the opposite travelling waves to a stationary wave.

Applying the second boundary condition eq. (3) to eq (4).

As Sin ωt ≠0 and Cos ωt ≠ 0,

Hence, $\text{Sin } k\ell = 0$,

which gives the general solution for angle *kl* to be

$\therefore \quad k\ell = n\pi$ where, n = 1,2, 3…..

As 'ℓ' is constant, k is limited to discrete set of values, known as eigen values.

$\therefore \quad k_n = \frac{n\pi}{\ell}$ where, n = 1, 2, 3, …. ------- (5)

$\therefore \nu_n = n\left(\frac{\upsilon}{2\ell}\right)$ where, n = 1, 2, 3, …. ------- (6)

Since, $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{\lambda\nu} = \frac{2\pi\nu}{\upsilon}$ $(\because \upsilon = \nu\lambda)$

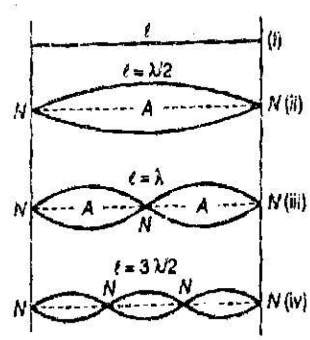
$\therefore \nu = \frac{k\upsilon}{2\pi}$

From eq. (5), $\nu = \frac{n\pi\upsilon}{2\pi\ell}$

$\nu = n\left(\frac{\upsilon}{2\ell}\right)$

From eq. (6) it is clear that the string can have a set of eigen or proper frequencies only. The equation represents modes of vibration corresponding to n^{th} harmonic frequency.

Different modes of vibration are shown in figure.



(ii) First mode of vibration (or) first harmonic

(iii) Second mode of vibration (or) Second harmonic, 1^{st} overtone

(iv) Third mode of vibration (or)

Third harmonic, 2nd overtone

Fundamental frequency corresponding to n =1 is,

$$\nu_1 = \left(\frac{\upsilon}{2\ell}\right)$$

$$\nu_1 = \frac{1}{2\ell}\sqrt{\frac{T}{m}} \quad \because \upsilon = \sqrt{\frac{T}{m}} \qquad \text{------------------(7)}$$

This is called first harmonic frequency.
The nth harmonic mode of frequency is,

$$\nu_n = \frac{n}{2\ell}\sqrt{\frac{T}{m}}$$

This is called (n–1) overtone.

OVERTONES AND HARMONICS:

(i) When the string is plucked at the middle, it vibrates with nodes(N) at the end and antinode (A) at the middle as shown in fig(ii). The frequency of vibration here is called the fundamental frequency (or) first harmonic.

The frequency, $\nu_1 = \frac{1}{2\ell}\sqrt{\frac{T}{m}}$ n = 1

(ii) If the string is vibrating in two segments as shown in fig (iii),

The frequency of vibration, $\nu_2 = \frac{2}{2\ell}\sqrt{\frac{T}{m}}$ n = 2

$$\nu_2 = 2\nu_1$$

This is called second harmonic (or) first overtone.

(iii) If the string is vibrating in three segments as shown in fig (iv),

The frequency of vibration, $\nu_3 = \frac{3}{2\ell}\sqrt{\frac{T}{m}}$ n = 3

$$\nu_3 = 3\nu_1$$

This is called third harmonic (or) second overtone.

(iv) If the string is vibrating in four segments, then

The frequency of vibration, $\nu_4 = \frac{4}{2\ell}\sqrt{\frac{T}{m}}$ n = 4

$$\nu_4 = 4\nu_1$$

This is called fourth harmonic (or) third overtone.

So, in case of stretched string the frequencies are in the ratio, $\nu_1 : \nu_2 : \nu_3 \ldots = 1 : 2 : 3 \ldots.$

Laws of Transverse Vibrations of Strings:

The fundamental frequency of vibrating string, $\nu = \frac{1}{2\ell}\sqrt{\frac{T}{m}}$

1st Law: $\nu \propto \frac{1}{\ell}$ when T, and m are constant.

i.e., the fundamental frequency of vibrating string is inversely proportional to the length of the string, when tension and linear density are constant.

2nd Law: $\nu \propto \sqrt{T}$ when l and m are constant.

i.e., the fundamental frequency of vibrating string is directly proportional to the square root of tension in the string, when length and linear density are constant.

3rd Law: $\nu \propto \frac{1}{\sqrt{m}}$ when ℓ and T are constant.

i.e, the fundamental frequency of vibrating string is inversely proportional to the square root of linear density of string, when tension and length of the string are constant.

PROBLEMS

1. A travelling wave propagates according to the expression y = 0.03 Sin (3x – 2t) where 'y' is the displacement at position 'x' at time 't'. Taking the units to be in S.I, determine (a) The amplitude (b) The wave length (c) The frequency and (d) The period of the wave.

Sol: we know that, $y = a \sin(kx - \omega t)$

The given equation is, $y = 0.03 \sin(3x - 2t)$

On comparing, (a) amplitude, a = 0.03 m

(b) wave length, $\lambda = \frac{2\pi}{3}$ $(\because k = 3)$

$\lambda = \frac{2 \times 3.14}{3}$ = 2.09 m

(c) Frequency, $\nu = \frac{\omega}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} (\because \omega = 2)$

$\nu = 0.31$ Hz

(d) Time period, $T = \frac{1}{\nu} = \pi = 3.14$ sec

2. Standing waves are produced by the superposition of two waves $y_1 = 10 \text{ Sin } (3\pi t - 4x)$ and $y_2 = 10 \text{ Sin } (3\pi t + 4x)$. Find the amplitude of motion at x = 18 ?

Sol: given that, $y_1 = 10 \text{ Sin } (3\pi t - 4x)$

$y_2 = 10 \text{ Sin } (3\pi t + 4x)$

The resultant displacement is given by, $y = y_1 + y_2$

$y = 10 \text{ Sin } (3\pi t - 4x) + 10 \text{ Sin } (3\pi t + 4x)$

$= 10 [\text{Sin } 3\pi t \text{ Cos } 4x - \text{Cos } 3\pi t \text{ Sin } 4x + \text{Sin } 3\pi t \text{ Cos } 4x + \text{Cos } 3\pi t \text{ Sin } 4x]$

$= 10 \text{ X } 2 \text{ Sin } 3\pi t \text{ Cos } 4x$

$y = 20 \text{ Sin } 3\pi t \text{ Cos } 4x = 20 \text{ Cos } 4x \text{ Sin } 3\pi t$

The amplitude of motion is $A = 20 \text{ Cos } 4x$

When x = 18, $4x = 4 \text{ X } 18 = 72 = 72 \text{ x } \frac{\pi}{3.14}$ rad = 22.9 π rad

$A = 20 \text{ Cos } (22.9\pi) = 20 \text{ x } (0.9673)$

A = 19.35 units of length.

3. A string vibrates according to the equation $y = 5 \text{ Sin } \left(\frac{\pi x}{3}\right) \text{Cos } 40\pi t$, where x, y are in cm, and t is in sec. Find the distance between two successive nodes and the speed of the particle of the string at position x = 1.5 cm when t = 9/8 sec?

Sol: At nodes y = 0, thus

$$0 = 5 \text{ Sin } \left(\frac{\pi x}{3}\right) \text{Cos } 40\pi t$$

As $\text{Cos } 40\pi t \neq 0$, $5 \text{ Sin } \left(\frac{\pi x}{3}\right) = 0$

$\frac{\pi x}{3} = n\pi$ where, n = 0, 1, 2, 3, ….

x = 3n = 0, 3, 6, 9, ….

So, the distance between two successive nodes = 3 cm

Speed of the particle, $\frac{\partial y}{\partial t} = -5\text{Sin}\left(\frac{\pi x}{3}\right) 40\pi \text{ Sin } 40\pi t + 5 \text{ Cos } 40\pi t \left(\frac{\pi}{3}\right) \text{ Cos } \left(\frac{\pi x}{3}\right)$

When x = 1.5 cm and t = 9/8 sec,

$$\frac{\partial y}{\partial t} = -5\text{Sin}\left(\frac{\pi \times 1.5}{3}\right) 40\pi \text{ Sin } \left(\frac{40\pi\, 9}{8}\right) + 5 \text{ Cos } 40\pi \left(\frac{9 \times \pi}{8 \times 3}\right) \text{ Cos } \left(\frac{\pi \times 1.5}{3}\right)$$

$$\frac{\partial y}{\partial t} = -5\mathrm{Sin}\left(\frac{\pi}{2}\right) \; 40\pi \; \mathrm{Sin}\; 45\pi \; + 5\;\mathrm{Cos}\; 40\pi \; \left(\frac{3\times\pi}{8}\right) \mathrm{Cos}\left(\frac{\pi}{2}\right)$$

$$\frac{\partial y}{\partial t} = 0 \qquad \because \mathrm{Cos}\left(\frac{\pi}{2}\right) = 0 \text{ and } \mathrm{Sin}\; 45\pi =$$

Hence, the particle is at rest at that position.

4. A steel wire 50 cm long has a mass of 5 gm. It is stretched with a tension of 400N. Find the frequency of the wire in fundamental mode of vibration?

Sol: given that, ℓ = 50 cm = 0.5 m

Mass = 5gm = 5 x 10^{-3} Kg

Tension, T = 400 N

Linear density, m = $\frac{5\times10^{-3}}{0.5}$ = 10^{-2} Kg/m

Frequency, $\nu = \frac{1}{2\ell}\sqrt{\frac{T}{m}} = \frac{1}{2\times 0.5}\sqrt{\frac{400}{10^{-2}}}$ = 1x 20x10

ν = 200 Hz.

5. The fundamental frequency of a stretched string of length 1m is 256 Hz. Find the frequency of the same string of half the original length under identical conditions?

Sol: $\nu \;\alpha\; \frac{1}{\ell}$

$\nu\ell$ = constant

$\therefore \nu_1\ell_1 = \nu_2\ell_2$

Given that, ν_1 = 256 Hz ν_2 = ?

ℓ_1 = 1 m ℓ_2 = 0.5 m

$\therefore$ 256 x 1 = 0.5 ν_2

$\therefore \nu_2 = \frac{256}{0.5}$ = 2 x 256 = 512 Hz.

6. Calculate the speed of transverse waves in a wire of 1mm^2 cross-section under the tension produced by 0.1 Kg weight. Specific gravity of material of wire is 9.81 gm/cm^3 and g = 9.81m/sec^2?

Sol: T = Mg = 0.1 x 9.81 = 0.981 N

Linear density, m = area of cross-section x Specific gravity

m = 10^{-6} x 9.81 x 10^3 = 9.81 x 10^{-3} Kg/m

since, area of cross-section = 1mm^2= 10^{-6} m^2

Specific gravity = 9.81 gm/cm^3 = 9.81 x 10^3Kg/m^3

Velocity, $\upsilon = \sqrt{\frac{T}{m}} = \sqrt{\frac{0.981}{9.81\times10^{-3}}} = \sqrt{\frac{9.81\times10^{2}}{9.81}} = 10$ m/s

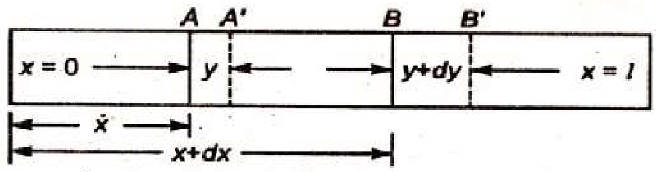
****

IV (B) VIBRATING BARS

Velocity of longitudinal waves in a bar:

Consider a bar of length ‘l’ of uniform cross-section ‘a’. The bar is made of homogeneous and isotropic material having a large length as compared to its area of cross section. The bar has only longitudinal vibrations and not transverse vibrations.

It is also assumed that at any given time, the displacement of all the particles at any cross-sectional area are the same.



As shown in figure, consider a small part ‘AB’ of length ‘dx’ of the bar in unstrained position at a distance x and x + dx. Under the influence of longitudinal waves, the planes A and B are displaced to new positions A^1 and B^1 respectively.

Let the displacement of plane A to A^1 is ‘y’ at any time when longitudinal wave passed through it.

The displacement of B to $B^1$is, y + dy .

$$y+dy=y+\left(\frac{\partial y}{\partial x}\right) dx$$ (Taylor’s series first two terms)

The longitudinal extension of the element is

$$(y + dy) - y= \left\{ y+\left(\frac{\partial y}{\partial x}\right) dx - y \right\} = \left(\frac{\partial y}{\partial x}\right) dx$$

$$\text{Longitudinal Strain} = \frac{Change\ \ in\ length}{original\ length} = \frac{\left(\frac{\partial y}{\partial x}\right) dx}{dx} = \left(\frac{\partial y}{\partial x}\right)$$

$$\text{Young's modulus, } Y = \frac{Stress}{Strain}$$

$$\text{Longitudinal Stress} = Y \text{ x Strain} = Y \left(\frac{\partial y}{\partial x}\right)$$

Force on the surface element at A = Longitudinal Stress X area of cross- section

$$= Y \left(\frac{\partial y}{\partial x}\right) \text{ x a}$$

$$= Y\, a \left(\frac{\partial y}{\partial x}\right)$$

Similarly, Force on the surface element at B = Y a $\frac{\partial}{\partial x}(y+dy)$

$$= \text{Ya}\frac{\partial}{\partial x}\left(y+\left(\frac{\partial y}{\partial x}\right)dx\right)$$

$$= \text{Y a}\left(\frac{\partial y}{\partial x}\right) + \text{Y a}\left(\frac{\partial^2 y}{\partial x^2}\right)dx$$

∴ The resultant force to which the elementary part is subjected

$$= \left\{\text{Y a}\left(\frac{\partial y}{\partial x}\right) + \text{Y a}\left(\frac{\partial^2 y}{\partial x^2}\right)dx\right\} - \text{Y a}\left(\frac{\partial y}{\partial x}\right)$$

$$= \text{Y a}\left(\frac{\partial^2 y}{\partial x^2}\right)dx \text{ --------(1)}$$

This restoring force tries to bring the displaced mass of elementary part to its mean position.

At the same time, it produces acceleration in it.

According to Newton's second law of motion,

Force on element 'dx' = mass x *acceleration*

Mass of the element = Volume X density

= a (dx) ρ where, ρ = density of the material

We know that, Acceleration = $\left(\frac{\partial^2 y}{\partial t^2}\right)$

∴ Force = a (dx) ρ $\left(\frac{\partial^2 y}{\partial t^2}\right)$ = a ρ $\left(\frac{\partial^2 y}{\partial t^2}\right)dx$ **------------ (2)**

From eq. (1) & (2) we get,

$$\text{a}\,\rho\left(\frac{\partial^2 y}{\partial t^2}\right)dx = \text{Y a}\left(\frac{\partial^2 y}{\partial x^2}\right)dx$$

$$\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{Y}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right) \textbf{------------ (3)}$$

The wave equation is given by, $\left(\frac{\partial^2 y}{\partial t^2}\right) = \upsilon^2 \left(\frac{\partial^2 y}{\partial x^2}\right)$ **------------ (4)**

Comparing eqs. (3) & (4) we get, $\upsilon^2 = \frac{Y}{\rho}$

or $\therefore \upsilon = \sqrt{\frac{Y}{\rho}}$ -------------(5)

This is the velocity of longitudinal wave in a bar.
It is clear from eq. (5) that velocity of longitudinal wave is
(i) directlyproportional *to the square root of longitudinal elasticity.*
(ii) inversely proportional to the square root of density of material, and
(iii) independent of shape and size of the cross-section.
(iv)

GENERAL SOLUTION OF LONGITUDINAL WAVE EQUATION:

The general solution of wave equation for the transverse vibrations of strings is applied to the longitudinal waves. Hence

$y = f_1(vt-x) + f_2(vt + x)$ -------------------(1)

Here y varies as a harmonic function of time, the simple harmonic solution isType equation here. be expressed as,

$y = a_1 \sin(\omega t - kx) + a_2 \sin(\omega t + kx) + b_1 \cos(\omega t - kx) + b_2 \cos(\omega t + kx)$ -------(2)

where a_1, a_2, b_1 and b_2 are amplitude constants.

We know that $k = \frac{2\pi}{\lambda} = \frac{\omega}{v}$

Where, k is the propagation constant, ω the angular frequency $(2\pi\,\vartheta)$ and v, the velocity of longitudinal waves.

Boundary Conditions: The following boundary conditions are applied,

(i) At a point where the bar is fixed, the displacement is zero at all time, i.e., y = 0 (at all time) ----(1)

(ii)At the free end, there can be no internal elastic force, hence, $\frac{dy}{dx} = 0$ at all time $\boldsymbol{\frac{dy}{dx}}$ **= 0(at all time)**

LONGITUDINAL VIBRATIONS OF A BAR RIGIDLY FIXED AT BOTH ENDS:

This is also known as fixed-fixed bar. When a bar is clamped at its ends, stationary waves are formed with antinode at the middle and node at the ends.

Boundary conditions are, y = 0 when x = 0 at any time 't'

and y = 0 when x= ℓ at any time 't' **---------(1)**

We know, the general solution of longitudinal wave is,

$y = a_1 \text{Sin}(\omega t - kx) + a_2 \text{Sin}(\omega t + kx) + b_1 \text{Cos}(\omega t - kx) + b_2 \text{Cos}(\omega t + kx)$ **--(2)**

Applying the first boundary condition, y = 0 when x = 0

$0 = a_1 \text{Sin}\,\omega t + a_2 \text{Sin}\,\omega t + b_1 \text{Cos}\,\omega t + b_2 \text{Cos}\,\omega t$

$0 = (a_1 + a_2) \sin \omega t + (b_1 + b_2) \cos \omega t$

As, $\sin \omega t \neq 0$ and $\cos \omega t \neq 0$

$a_1 + a_2 = 0$ and $b_1 + b_2 = 0$

Thus, $a_1 = -a_2$, $b_1 = -b_2$ **------- (3)**

Substituting eq. (3) in eq. (2)

$y = a_1 [\sin (\omega t - kx) - \sin (\omega t + kx)] + b_1 [\cos (\omega t - kx) - \cos (\omega t + kx)]$

$y = a_1 [(\sin \omega t \cos kx - \cos \omega t \sin kx) - (\sin \omega t \cos kx + \cos \omega t \sin kx)] +$

$b_1 [(\cos \omega t \cos kx + \sin \omega t \sin kx) - (\cos \omega t \cos kx - \sin \omega t \sin kx)]$

$y = a_1[-2\cos \omega t \sin kx] + b_1[2\sin \omega t \sin kx]$

$y = (-2a_1\cos \omega t + 2b_1 \sin \omega t) \sin kx$

$y = (A \cos \omega t + B \sin \omega t) \sin kx$-----------(4)

where, $A = -2a_1$ and $B = 2b_1$

Now apply boundary condition $y = 0$ when $x = \ell$

$0 = (A \cos \omega t + B \sin \omega t) \sin k\ell$

Since, A & $B \neq 0$, (otherwise there will be no wave),

Hence, $\sin k\ell = 0$

$\therefore \quad k\ell = n\pi$ where, n = 1,2, 3...

n =0 isnot taken as it corresponds to the condition of no wave (or a wave of infinite length).

Replacing k by k_n (because of dependence of k on the integer), thus

or, $k_n = \dfrac{n\pi}{\ell}$ where, n = 1, 2, 3, ….. ------- (5)

This equation shows only certain modes of vibration are allowed.

The frequency of allowed modes of vibration are given by,

$$\frac{\omega_n}{\upsilon} = \frac{n\pi}{\ell}$$

Since, $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi\nu}{\lambda\nu} = \dfrac{2\pi\nu}{\upsilon} = \dfrac{\omega}{\upsilon}$

$(\because \upsilon = \nu\lambda \ \& \ \omega = 2\pi\nu)$

$\omega_n = \dfrac{n\pi\upsilon}{\ell}$ where, n = 1, 2, 3, …..

$2\pi \nu_n = \dfrac{n\pi\upsilon}{\ell}$

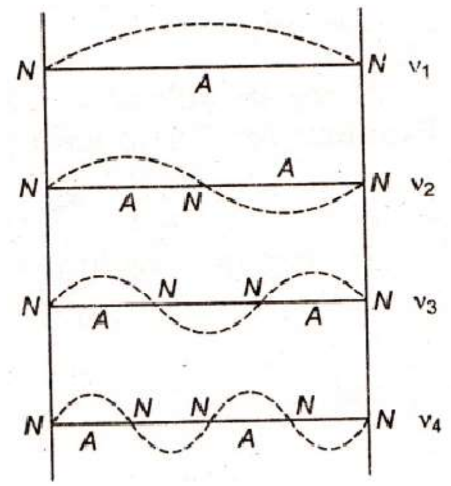
$$\nu_n = \frac{n\upsilon}{2\ell} \qquad n = 1, 2, 3, \ldots$$

We know, $\upsilon = \sqrt{\frac{Y}{\rho}}$

$$\therefore \nu_n = \frac{n}{2\ell}\sqrt{\frac{Y}{\rho}}$$

In the fundamental mode of vibration, the two ends of the rod are nodes and only one antinode at the midpoint. The higher harmonics are in the ratio 1:2:3: …

The various modes vibration are shown in figure.



The complete solution of longitudinal wave is

$$y = \sum_{n=1}^{\infty}\left(A_n Cos\,\omega\,t + B_n Sin\,\omega\,t\right) Sin\,k_n x$$

Modes of vibrations of fixed- fixed bar

LONGITUDINAL VIBRATIONS OF A BAR CLAMPED AT THE MIDDLE:

When a bar is clamped at its middle point, stationary waves are formed with node at the middle and antinode at the ends.

The boundary conditions are,

$\frac{\partial y}{\partial x} = 0$ when x = 0 for all time 't'

And $y = 0$ when $x = \ell/2$ for all time 't' -------(1)

The general solution of longitudinal wave is,

$y = a_1 \, Sin\,(\omega t - kx) + a_2 \, Sin\,(\omega t + kx) + b_1 \, Cos\,(\omega t - kx) + b_2 \, Cos\,(\omega t + kx)$ ------------(2)

Now,

$\frac{\partial y}{\partial x} = -ka_1 \, Cos\,(\omega t - kx) + ka_2 \, Cos\,(\omega t + kx) + kb_1 \, Sin\,(\omega t - kx) - kb_2 \, Sin\,(\omega t + kx)$

Apply boundary condition, $\frac{\partial y}{\partial x} = 0$ when x = 0

$0 = -ka_1 \, Cos\,\omega t + ka_2 \, Cos\,\omega t + kb_1 \, Sin\,\omega t - kb_2 \, Sin\,\omega t$

$0 = k \, Cos\,\omega t \, (a_2 - a_1) + k \, Sin\,\omega t \, (b_1 - b_2)$

As, $Sin\,\omega t \neq 0$ and $Cos\,\omega t \neq 0$

$a_1 = a_2, \quad b_1 = b_2$ ------- (3)

substituting these values in (2) we get,

$$y = a_1 [\text{Sin}(\omega t - kx) + \text{Sin}(\omega t + kx)] + b_1 [\text{Cos}(\omega t - kx) + \text{Cos}(\omega t + kx)]$$

$$y = a_1 [\text{Sin}\,\omega t\,\text{Cos}\,kx - \text{Cos}\,\omega t\,\text{Sin}\,kx + \text{Sin}\,\omega t\,\text{Cos}\,kx + \text{Cos}\,\omega t\,\text{Sin}\,kx]$$

$$+ b_1 [\text{Cos}\,\omega t\,\text{Cos}\,kx + \text{Sin}\,\omega t\,\text{Sin}\,kx + \text{Cos}\,\omega t\,\text{Cos}\,kx - \text{Sin}\,\omega t\,\text{Sin}\,kx]$$

$$y = a_1[2\text{Sin}\,\omega t\,\text{Cos}\,kx] + b_1[2\,\text{Cos}\,\omega t\,\text{Cos}\,kx]$$

$$y = (2a_1\text{Sin}\,\omega t + 2b_1\,\text{Cos}\,\omega t)\,\text{Cos}\,kx$$

$$y = (A\,\text{Cos}\,\omega t + B\,\text{Sin}\,\omega t)\,\text{Cos}\,kx \text{ -----------(4)}$$

where, $A = 2b_1$ and $B = 2a_1$

Apply boundary condition, $y = 0$ when $x = \frac{\ell}{2}$

$$0 = (A\,\text{Cos}\,\omega t + B\,\text{Sin}\,\omega t)\,\text{Cos}\frac{k\ell}{2} \text{ ----------(5)}$$

Since, A & $B \neq 0$, $\text{Cos}\frac{k\ell}{2} = 0$

$$\frac{k\ell}{2} = \frac{(2n-1)\pi}{2} \quad \text{where, n= 1,2,3, …}$$

These are allowed vibrations in case of a bar clamped at the middle,

Hence, $k = \frac{(2n-1)\pi}{\ell}$

Considering the dependence of k on integer, we have

or, $K_n = \frac{(2n-1)\pi}{\ell}$ where, n= 1,2,3,…

The frequency, $\frac{\omega_n}{\upsilon} = K_n$ Since, $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{\lambda\nu} = \frac{2\pi\nu}{\upsilon} = \frac{\omega}{\upsilon}$

$$\frac{\omega_n}{\upsilon} = \frac{(2n-1)\pi}{\ell}$$

$$\omega_n = \frac{(2n-1)\pi\upsilon}{\ell} \qquad n= 1,2,3,…$$

$$\omega_n = 2\pi\,\nu_n$$

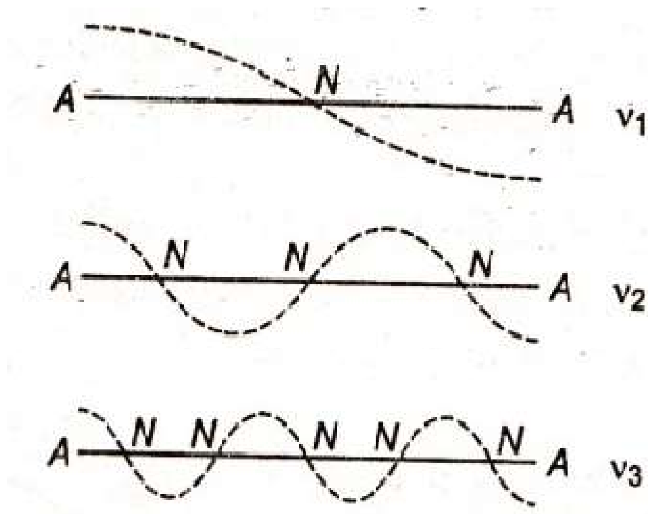
Therefore, $2\pi\,\nu_n = \frac{(2n-1)\pi\upsilon}{\ell}$

$$\nu_n = \frac{(2n-1)\upsilon}{2\ell} \text{ ------(6)}$$

The frequency of n^{th} mode is,

$$\nu_n = \frac{(2n-1)}{2\ell}\sqrt{\frac{Y}{\rho}} \quad \text{-----(7)} \qquad \text{Since, } \upsilon = \sqrt{\frac{Y}{\rho}}$$

The modes of vibration are shown in figure. The odd harmonics are produced while even harmonic is completely absent.
Frequency are in the ratio 1: 3: 5: …



LONGITUDINAL VIBRATIONS OF A BAR FIXED AT ONE END AND FREE AT THE OTHER:

This is also known as the fixed-free bar. The boundary conditions are

y = 0 at x =0 for all time t

$\frac{\partial y}{\partial x}$ = 0 at x = l for all time t ---------------(1)

The general solution of longitudinal wave is,

$y = a_1 \text{Sin}(\omega t - kx) + a_2 \text{Sin}(\omega t + kx) + b_1 \text{Cos}(\omega t - kx) + b_2 \text{Cos}(\omega t + kx)$ ------------(2)

Applying the first boundary condition, we have

$0 = a_1 \sin \omega t + a_2 \text{Sin}\omega t + b_1 \text{Cos}\omega t + b_2 \text{Cos}\omega t$

$0 = (a_1 + a_2) \text{Sin } \omega t + (b_1 + b_2) \text{Cos } \omega t$

$\therefore a_1 = -a_2,$ and $b_1 = -b_2$

Substituting these values in eq. (2) we get

$y = a_1[\text{Sin}(\omega t - kx) - \text{Sin}(\omega t + kx)] + b_1[\text{Cos}(\omega t - kx) - \text{Cos}(\omega t + kx)]$

$y = a_1[\sin \omega t \text{Cos} kx - \text{Cos}\omega t \text{Sin } kx - \text{Sin}\omega t \text{Cos } kx - \text{Sin}\omega t \text{Cos} kx - \text{Cos}\omega t \text{Sin } kx]$

$+ b_1[\text{Cos}\omega t \text{Cos} kx + \sin \omega t \text{Sin } kx - \text{Cos}\omega t \text{Cos } kx + \text{Sin}\omega t \text{Sin } kx]$

$y = a_1[-2\text{Cos}\omega t \text{Sin } kx] + b_1[2 \sin \omega t \text{Sin } kx]$

Let $-2a_1 = A$ and $2b_1 = B$

$\therefore \quad y = (A \text{Cos}\omega t + B \sin \omega t) \text{Sin } k x$-----------------(3)

Now, applying the boundary conditions $\frac{\partial y}{\partial x} = 0$ at $x = l$.

Differentiating eq. (3) with respect to x.

Hence, $\frac{\partial y}{\partial x}$ = (A Cos ωt + B sin $\omega t)k$ cos k l

or 0 = (A Cos ωt + B sin $\omega t)k$ cos k l

cos k l= 0   Since, A and B ≠ 0--------------(4)

The allowed frequencies should satisfy k l = (2n-1)$\frac{\pi}{2}$ where n = 1,2, 3…

Replacing k by k_n, we get

k_n = (2n-1)$\frac{\pi}{2l}$ n = 1,2, 3….

ω_n = (2n-1)$\frac{\pi\vartheta}{2l}$ n = 1,2, 3…. -----------------(5)

$\vartheta_n = \frac{(2n-1)\vartheta}{4l}$ n = 1,2, 3….

From eq. (5), it is clear that

(i) Only odd harmonics are present in a fixed- free bar
(ii) The fundamental frequency is half that of a free-free bar
(iii) The quantity of sound is altereddue to the absence of even harmonics

The complete longitudinal wave solution, in respected of a fixed-fixed bar, may be considered as sum of n harmonic solution, i.e., may be considered as the sum of n harmonics.

`

PROBLEMS:

1.The density of aluminium is 2.8×10^3 Kg/m^3 and its Young's modulus is 7×10^{10} pascals. If the frequency of the Aluminium rod is 500Hz, Calculate the velocity of sound and wavelength through the rod?

Sol: given that, $\rho = 2.8 \times 10^3$ Kg/m^3

$Y = 7 \times 10^{10}$ pascals

$\nu = 500$ Hz $\lambda = ?$ and $\upsilon = ?$

Velocity of longitudinal wave $\upsilon = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{7\times 10^{10}}{2.8\times 10^{3}}} = \frac{10^4}{2}$

$\upsilon = 5 \times 10^3$ m/s

$$\lambda = \frac{\upsilon}{\nu} = \frac{5\times 10^{3}}{500} = 10 \text{ m}$$

2. A copper rod of length 4m is free at its ends, the diameter of the cross section of the rod is 0.01m. Find the fundamental frequency of the longitudinal vibrations of the rod? (velocity of sound in copper is 3560m/s)

Sol: The frequency, $\nu = \frac{(2n-1)\upsilon}{2\ell}$

For fundamental frequency n =1, $\nu = \frac{\upsilon}{2\ell}$

$\upsilon = 3560$m/s and $\ell = 4$ m

$$\nu = \frac{3560}{2\times 4} = 445 \text{ Hz}$$

3. A steel rod of length one meter and density 7.1 gm/cc is clamped at itsmiddle and longitudinal vibrations are set up in it. If the fundamental frequency is 2600 Hz. Find the velocity of sound in the rod and Young's modulus of material of the rod?

Sol: given that,

$\rho = 7.1$gm/cc $= 7.1 \times \frac{10^{-3}}{10^{-6}}$ Kg/m^3 $= 7.1 \times 10^3$ Kg/m^3

$\ell = 1$ m and $\nu = 2600$ Hz

$\nu = \frac{\upsilon}{2\ell}$ or velocity $\upsilon = 2\nu\ell = 2 \times 2600 \times 1 = 5200$ m/s

$$\upsilon = \sqrt{\frac{Y}{\rho}} \text{ or } Y = \rho\,\upsilon^2 = 7.1 \times 10^3 \times 5200 \times 5200$$

$$Y = 19.2 \times 10^{10} \text{ N/m}^2$$

4. A brass rod of length one meter is clamped at its middle point. If it is madeto vibrate longitudinally, find the fundamental frequency and frequencies of first two overtones?

($Y = 10 \times 10^{10}$ N/ m^2 and $\rho = 8.3 \times 10^3$ Kg/m^3)

Sol: given that, $\ell = 1$ m

$Y = 10 \times 10^{10}$ N/ m^2

$\rho = 8.3 \times 10^3$ Kg/m^3

The frequency, $\nu = \frac{(2n-1)}{2\ell}\sqrt{\frac{Y}{\rho}}$

For fundamental frequency n =1, $\nu = \frac{1}{2\ell}\sqrt{\frac{Y}{\rho}}$

$$\nu = \frac{1}{2}\sqrt{\frac{10\times10^{10}}{8.3\times10^{3}}}$$

$$\nu = \frac{1}{2}\sqrt{\frac{10}{8.3}} \times 10^4 = \frac{0.3471}{2} \times 10^4$$

$$\nu = 1735.5 \text{ Hz}$$

First overtone, $\nu_1 = 3\nu = 3 \times 1735.5 = 5206.5$ Hz (n=2)

Second overtone, $\nu_2 = 5\nu = 5 \times 1735.5 = 8677.5$ Hz (n= 3)

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UNIT-V

ULTRASONICS

Introduction:

Sound is produced from vibrating bodies. Sound waves are longitudinal mechanical waves. The frequency range of these waves is very high.

The frequencies of sound between **20 Hz to 20,000 Hz** are called **audiable frequencies.** The human ear can recognize these sounds only.

The frequencies of sound below **20 Hz** are called **Infrasonic**. Human ear cannot recognize these sounds. The wavelength of these waves is more.

The frequencies of sound more than **20,000 Hz** are called **Ultrasonics**. Human ear cannot recognize these sounds also. The wavelength of these waves is less, it is about less than 1.8cm. Hence, they can travel in a specific direction.

PROPERTIES OF ULTRASONICS:

1.Ultrasonics are highly energetic.

2. Their speed of propagation increases with frequency.

3. They show negligible diffraction due to their small wavelength. Hence, they can be transmitted over long distances without any appreciable loss of energy.
4. Intense Ultrasonic radiation has a disruptive effect on liquid by causing bubbles to be formed.
5.When Ultrasonic waves are propagated in liquid bath a plane diffraction grating is formed, which can diffract light.

(When Ultrasonic waves are propagated in liquid bath, stationary wave pattern is formed due to the reflection of the wave from the other end. The density of the liquid thus varies from layer to layer along the direction of propagation. In this way a plane diffraction grating is formed which can diffract light.)

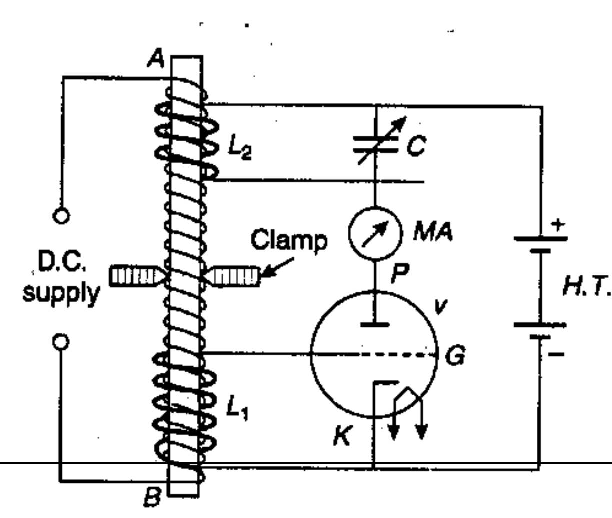
PRODUCTION OF ULTRASONICS:

Ultrasonics can be produced in two important methods

1. Magnetostriction method
2. Piezo-electric method

Magnetostriction method is used to produce Ultrasonics of frequencies up to 100 KHz. For the production of Ultrasonics of frequencies more than 100 KHzPiezo-electric method is used.

1. MAGNETOSTRICTION METHOD:



Magnetostriction:

When a rod of ferromagnetic material such as Iron or Nickel, is placed in a magnetic field parallel to its length, a small expansion is occurred, this phenomenon is called **Magnetostriction.** This change in length depends on magnitude of the field and nature of the material.

If the rod is placed inside a coil carrying an alternating current, then it suffers the same change in length for each half cycle alternating current. This results in setting up vibrations in the rod whose frequency is twice that of alternating current. However, if the frequency of the a.c. is the same as the natural frequency of the rod, then resonance occurs and the amplitude of vibration is considerably increased. Sound waves are emitted from the ends of the rod. More over if the applied frequency is the order of Ultrasonics frequency, the rod sends out Ultrasonic waves.

Procedure: An experimental arrangement to produce Ultrasonic waves is shown in figure. The rod is permanently magnetized by passing direct current (d.c.) in the coil which is wrapped round the rod. There are two coils L_1 and L_2 which are also wrapped round the rod as shown in figure. The coil L_2 is connected in the plate circuit of valve V, while L_1 is connected in the grid circuit. A variable condenser C is connected across the coil L_2, a milli ammeter (mA) reads plate current.

As the internal diameter of the coils is more, the rod can freely produce longitudinal vibrations. The values of the Inductance of the coil L_2 and the capacity of the variable condenser C decides the frequency of the electric oscillator. When the frequency of the electric oscillator coincides with the natural frequency of the rod then resonance occurs and the rod vibrates with maximum amplitude and produces Ultrasonics. By varying the length of the rod and capacity of the variable condenser C we can produce the Ultrasonics of required frequency.

The velocity of Ultrasonics in the rod is (υ) $\upsilon = \sqrt{\frac{Y}{\rho}}$

Where Y = Young's modulus of the rod

ρ = density of the material of the rod

If 'ℓ' is the length of the rod, then the fundamental wave length becomes 2 ℓ

Hence, the frequency $\nu = \frac{\upsilon}{2\ell}$ or $\nu = \frac{1}{2\ell}\sqrt{\frac{Y}{\rho}}$

In this method Ultrasonic waves having less frequency were produced.

PIEZO-ELECTRIC EFFECT:

When certain crystals likequartz, tourmaline etc are stretched or compressed along certain axis, an electric potential difference is produced along a perpendicular axis, this is called Piezo-electric effect.

The converse of this effect is also true, i.e. when an alternating potential difference is applied along the electric axis, the crystal is set into elastic vibration along the mechanical axis.

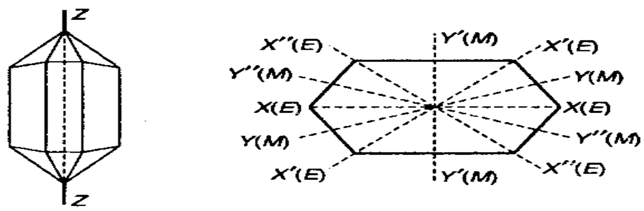


Fig-1

Quartz crystal is six-sided prism with pyramid shaped ends as shown in figure-1 has following three major axes.

a) **Optic axis or Z-axis**: The line joining the apexes of the end pyramids is known as Z-axis.
b) **The electric axis or X-axis:** The axis passes through any set of opposite corners known as X-axis.
c) **The mechanical axis or Y-axis:** The axis passes through the opposite faces known as Y-axis.

The X-cut and Y-cuts of the crystal are shown in figure-2.

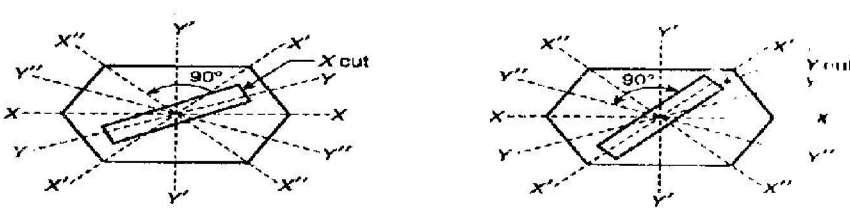


Fig-2

The X-cut slab makes an angle 90^o with the X-axis while Y-cut slab makes an angle 90^o with the Y-axis. X-cut slabs are used for the generation of Ultrasonics, because they produce longitudinal waves. Y-cut slabs produce shear waves which can travel only in solids.

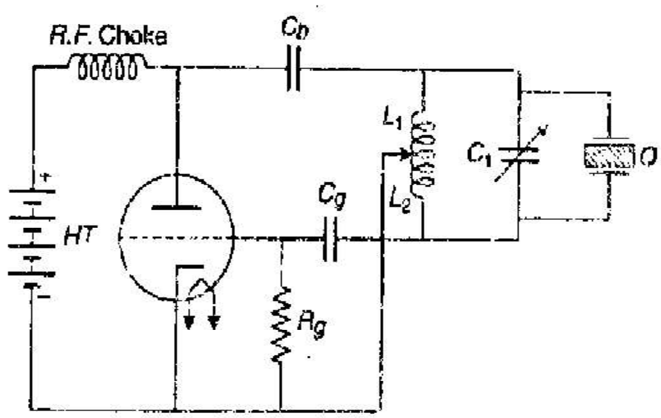
2. PIEZO-ELECTRIC METHOD:

Piezo-electric effect:

When certain crystals likequartz, tourmaline etc are stretched or compressed along certain axis, an electric potential difference is produced along a perpendicular axis, this is called Piezo-electric effect.

The converse of this effect is also true, i.e. when an alternating potential difference is applied along the electric axis, the crystal is set into elastic vibration along the mechanical axis. If the frequency of electric oscillations coincides with the natural frequency of the crystal, the vibrations will be of large amplitude. This phenomenon is used for the production of Ultrasonic waves. The alternating potential difference is obtained by a valve oscillator.

X-cut slabs are used for the generation of Ultrasonics, because they produce longitudinal waves.



C_b = blocking capacitor

C_g = grid condenser

R_g = grid leak resistor

C_1 = variable condenser

Q = X-cut quartz crystal

Description:

The experimental arrangement is shown in figure. The high frequency alternating voltage which is applied to crystal is obtained by Hartley Oscillatory circuit. The Hartley circuit consists of tuned circuit i.e. Inductance (L_1) and variable condenser (C_1) in parallel. One end of the tuned circuit is connected to the plate of a valve while the other is connected to the grid. The coil (L_1) is trapped at the centre and joined to the cathode. The X-cut quartz crystal Q is connected parallel to variable condenser C_1.

Procedure:

The proper grid bias is obtained by means of grid leak resistor Rg and grid condenser Cg. The d.c. voltage is applied to the plate through radio frequency choke. The radio frequency choke prevents the radio frequency current to pass through high-tension battery. C_b is the blocking capacitor which prevents the direct current to pass through the tank circuit, while by passes the radio frequency currents. The capacity of the variable condenser C_1 is adjusted so that the frequency of the oscillating circuit is tuned to the natural frequency of the crystal. Now the quartz crystal is set into mechanical vibrations and Ultrasonic waves are produced. The Ultrasonics of frequency 500 KHz are produced by this method. However, the frequency up to 15x 10^7 Hz can be produced by using tourmaline crystal.

The velocity of the quartz along X-direction is (υ)

$$\upsilon = \sqrt{\frac{Y}{\rho}}$$

Where Y = Young's modulus of crystal

ρ = density of crystal

If 't' is the thickness of the quartz slab in meters,

$$\upsilon = \nu \lambda = \nu (2t) \qquad \text{since, } \lambda = 2t$$

$$\nu = \frac{\upsilon}{2t}$$

$$\nu = \frac{1}{2t}\sqrt{\frac{Y}{\rho}}$$

By adjusting the variable capacitor C_1 of tank circuit, the crystal is made to vibrate at its natural frequency, then the frequency of oscillatory circuit gives the frequency of vibrations of quartz crystal.

Thus, $$\nu = \frac{1}{2\pi\sqrt{L_1 C_1}}$$

DETECTION OF ULTRASONICS:

1. Piezo - electric detector:

The quartz crystal can be used for the detection of Ultrasonics. One pair of faces of quartz crystal is subjected to Ultrasonics, on other faces which are perpendicular to the previous one varying electric charge are produced. These charges are very small. Hence, they are amplified and then detected by suitable means.

2. Kundt's tube:

A Kundt's tube can be used to detect Ultrasonics of relatively large wavelength. When Ultrasonics are passed through the tube, the lycopodium powder sprinkled in the tube collects in the form of heaps at the nodal points and it is blown off at the antinodal points.

3. Sensitive flame method:

When a sensitive flame is moved in a medium where Ultrasonics are present, the flame remains stationary at antinodes and flickers at nodes.

4.Thermal detector method:

In this method a fine platinum wire is moved in the medium of Ultrasonics, the temperature of the medium changes due to alternative compressions and rarefactions. There is a change of temperature at nodes while at antinodes the temperature remains constant. Hence the resistance of platinum wire changes at nodes and remains constant at antinodes. The changes in the resistance of platinum wire with respect to time can be detected by using sensitive bridge arrangement. The bridge will be in the balanced position when the platinum wire is at antinodes.

APPLICATIONS OF ULTRASONIC WAVES:

1. Detection of Submarines, Iceberg and other objects in Ocean:

A sharp Ultrasonic beam is directed in various directions into the sea. The reflection of waves from any direction shows the presence of some reflecting body in the Sea.

2.Depth of the Sea (Sonar- Sound Navigation and Ranging):

We know that Ultrasonic waves are highly energetic and show a little diffraction effect, hence they can be used to find the depth of the sea. The time interval between sending wave and the reflected wave from the sea is recorded. As the velocity of the wave is known, the depth of the sea can be estimated.

3. Cleaning and Clearing:

The ultrasonic waves can be used for cleaning utensils, washing clothes, removing dust and soot from the chimney.

4. Direction Signalling:

The ultrasonic waves can be concentrated into sharp beam due to smaller wavelength and hence they can be used for signalling in a particular direction.

5. Soldering and metal cutting:

Ultrasonic waves can be used for drilling and cutting process in metals. These waves can also be used for soldering.

Ex: Aluminium cannot be soldered by normal methods. For solder aluminium ultrasonic waves along with electrical soldering iron was used. Ultrasonic welding can be done at the room temperature.

6. Ultrasonic mixing:

Emulsion of two non- miscible liquids like oil and water can be formed by simultaneously subjecting to Ultrasonic radiations. Now a days most of the emulsion like polishes, paints, food products and pharmaceutical preparations are prepared by using ultrasonic mixing.

7. Destruction of lower life:

The animals like rats, frogs, fishes etc can be killed or injured by using high intensity Ultrasonics.

8. Treatment of neuralgic pain:

The body parts affected due to neuralgic or rheumatic pains on being exposed to Ultrasonics gets great relief from pain.

9. Detection of Abnormal growth:

Abnormal growth in the brain, certain tumours which cannot be detected by X-rays can be detected by using ultrasonic waves.

10.Ultrasonics in metallurgy:

To irradiate molten metals which are in the process of cooling so as to refine the grain size and to prevent the formation of cores and to release trapped gases the ultrasonic waves are used.

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Problems:

1. A quartz crystal of thickness 0.005 metre is vibrating at resonance. Calculate the fundamental frequency. Given Y for quartz = 7.9 x 10^{10} newton and ρ for quartz = 2650 kg/m^3.

Sol: we know $\upsilon = \sqrt{\frac{Y}{\rho}}$

Substituting the given values, we get

$$\upsilon = \sqrt{\frac{7.9 \times 10^{10}}{2650}}$$

υ = 5461 m/sec

For the fundamental mode of vibration, thickness t = $\frac{\lambda}{2}$

$$\lambda = 2t = 2 \times 0.005 = 0.01 \text{ m}$$

Now, $\upsilon = \nu \lambda$ or $\nu = \frac{\upsilon}{\lambda}$

$$\nu = \frac{5461}{0.01}$$

∴ $\nu = 0.5461 \times 10^6$ Hz

2. A piezo- electric crystal with vibrating length (t) = 3x 10^{-3} m having density (ρ) = 3.5x10^3 kg/m^3. If it is made of material of young's modulus (Y) = 8x10^{10}N/m^3, what is its fundamental frequency?

Sol: The fundamental frequency is given by

$$\nu = \frac{1}{2t}\sqrt{\frac{Y}{\rho}}$$

Substituting the given values, we get,

$$\nu = \frac{1}{2\times(3\times10^{-3})}\sqrt{\frac{8\times10^{10}}{3.5\times10^{3}}}$$

$$\nu = \frac{2\sqrt{2}}{6\times10^{-3}}\sqrt{\frac{10^8}{35}}$$

$$\nu = \frac{2\sqrt{2}\times10^{4}}{6\times10^{-3}\times5.916}$$

$$\nu = \frac{20\sqrt{2}\times 10^{6}}{6\times 5.916}$$

$$\nu = \frac{20\times 1.414\times 10^{6}}{6\times 5.916}$$

$\nu = 0.7967 \times 10^6$ Hz

$\nu = 0.7967$ MHz

3.A piezo- electric crystal has a thickness 0.002m. if the velocity sound wave in crystal is 5750m/s, calculate the fundamental frequency of the crystal.

Sol: For the fundamental mode of vibration, thickness t = $\frac{\lambda}{2}$

$\lambda = 2\,t = 2 \times 0.002 = 0.004$ m

$\upsilon = 5750$ m/s

$$\nu = \frac{\upsilon}{\lambda}$$

$$\therefore \nu = \frac{5750}{0.004}$$

$\nu = 1.4375 \times 10^6$ Hz

$\nu = 1.4375$MHz

4. Calculate the capacitance to produce ultrasonic waves of 106 Hz with an inductance of 1 Henry.

Sol: The frequency of LC circuit is given by,

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

$$\therefore C = \frac{1}{4\pi^2\nu^2 L}$$

$$C = \frac{1}{4(3.14)^2\times(10^6)^2\times 1}$$

$C = 0.025 \times 10^{-12}$ F

$C = 0.025$ PF (1PF = 10^{-12}F)

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