Classes of Operators on Hilbert Spaces
Extended Lecture Notes

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1 Basic Definitions and Examples

We shall continue to denote by $B(H)$ the set of all bounded operators acting on a complex Hilbert space H. For $T \in B(H)$ let $R(T) = \overline{T(H)}$. I will denote the identity operator acting on H.

Algebraic properties inside $B(H)$ endowed with the $*$ operation lead to striking analytic spectral properties.

numbers	functions	operators
complex, $z\overline{z} = \overline{z}z$	complex	normal, $T^*T = TT^*$.
real, $z = \overline{z}$	real	self-adjoint, $T = T^*$.
positive, $z\overline{z}$	positive	positive, T^*T
complex unit	into unit circle	unitary, $T^*T = TT^* = I$
$\{0, 1\}$	indicator function	projection, $T = T^2 = T^*$

Recall that there is a bijection between bounded operators on H and bounded conjugate linear bilinear forms on H given by

$$T \in B(H) \to B_T, \text{ where } \quad B_T(x, y) = (Tx, y) \qquad x, y \in H\,.$$

In fact, there is a bijection between operators acting on H and quadratic forms on H given by the assignment

$$T \in B(H) \to Q_T, \quad \text{where } Q_T(x) = (Tx, x) \quad x \in H\,.$$

This is the content of the following proposition.

1.1. Proposition. *For $T_1, T_2 \in B(H)$ we have that $Q_{T_1} = Q_{T_2}$ if, and only if, $T_1 = T_2$.*

Proof: Polarization identity for $T \in B(H)$ gives us

$$\begin{aligned}(Tx, y) &= (T(x + y), x + y) - (T(x - y), x - y) \\ &\quad + i(T(x + iy), x + iy) - i(T(x - iy), x - iy)\,.\end{aligned}$$

Consequently, $(T_1x, x) = (T_2x, x)$ for all $x \in H$ implies that $B_{T_1} = B_{T_2}$ and so $T_1 = T_2$. $\square$

1.2. Definition. An operator $T \in B(H)$ is called

(i) *normal* if $TT^* = T^*T$.

(ii) *self-adjoint* if $T = T^*$.

(iii) *positive* if $(Tx, x) \geq 0$ for all $x \in H$.

(iv) *unitary* if $T^*T = TT^* = I$

(v) *projection* if $T = T^2 = T^*$.

1.3. Proposition. *Let $T \in B(H)$. Then*

(i) *T is normal if, and only if, $\|Tx\| = \|T^*x\|$ for all $x \in H$.*

(ii) *T is self-adjoint if, and only if, (Tx, x) is real for all $x \in H$.*

(iii) *T is unitary if, and only if, T is an inner product preserving surjection.*

Proof: (i) For all $x \in H$ we have

$$(T^*Tx, x) - (TT^*x, x) = (Tx, Tx) - (T^*x, T^*x) = \|Tx\|^2 - \|T^*x\|^2 .$$

This, together with Proposition 1.1, implies (i).
(ii) For all $x \in H$ we have

$$(Tx, x) - (T^*x, x) = (Tx, x) - (x, Tx) = 2i\,\Im(Tx, x).$$

It means that $T = T^*$ if, and only if, (Tx, x) is real for all $x \in H$.
(iii) If T is unitary, then, for $x, y \in H$,

$$(Tx, Ty) = (x, T^*Tx) = (x, x) .$$

If $x \in R(T)^\perp$, then

$$0 = (TT^*x, x) = (x, x)$$

and so $x = 0$. Hence, T is a surjection which preserves the inner product. On the other hand, if T is an inner product preserving surjection, then T has an inverse and

$$(T^*Tx, x) = (Tx, Tx) = (x, x) ,$$

which implies that $T^{-1} = T^*$ and so $T^*T = TT^* = I$.

1.4. Example. (Discrete diagonal) Let H be a Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $g : \mathbb{N} \to \mathbb{C}$ be a bounded function. Define

$$Tx = \sum_{n=1}^{\infty} g(n)\,(x, e_n)e_n\,.$$

(This is a diagonal operator uniquely given by $Te_n = g(n)\,e_n$.)
Let us observe that

$$\|T\| = \sup_{n\in\mathbb{N}} |g(n)|\,.$$

As $(Te_n, e_n) = (e_n, \overline{g(n)}e_n)$ we see that

$$T^*x = \sum_{n=1}^{\infty} \overline{g(n)}\,(x, e_n)e_n$$

for all $x \in H$. All diagonal operators are normal. In a certain sense the converse holds - see below.

- T is self-adjoint $\Longleftrightarrow$ $g(n) = \overline{g(n)}$ for all $n \in \mathbb{N}$ (g is real).

- T is positive $\Longleftrightarrow$ $(Tx, x) \geq 0$ for all x $\Longleftrightarrow \sum_n g(n)|(x, e_n)|^2 \geq 0$ for all $x \in H$. But this is equivalent to $g(n) \geq 0$ for all $n \in \mathbb{N}$.

- T is unitary $\Longleftrightarrow$ $T^*T = TT^* = I$.

But $T^*Tx = \sum_{n=1}^{\infty} \overline{g(n)}g(n)(x, e_n)\,e_n = \sum_{n=1}^{\infty} |g(n)|^2\,(x, e_n)e_n$.
In other words, T is unitary $\Longleftrightarrow$ $|g(n)| = 1$ for all n.

Now we shall deal with a generalization of this example to cover also operators with continuous spectrum.

1.5. Example. Let (X, μ) be a σ-finite measure space. For a measurable function, f, on X define

$$\begin{aligned} \|f\|_\infty &= \inf\{K \geq 0 \,|\, |f(x)| \leq K \text{ for a.a. } x \in X\} \\ &= \sup\{L \geq 0 \,|\, |f| > L \text{ on some set of nonzero measure}\}\,. \end{aligned}$$

$L^\infty(X, \mu)$ ... space of all measurable function with finite $\|\cdot\|_\infty$. Put $H = L^2(X, \mu)$ and fix $f \in L^\infty(X, \mu)$.
Define multiplication operator M_f acting on H by

$$M_f(g) = f\,g\,, \quad g \in L^2(X, \mu)\,.$$

Observe that

$$\int_X |M_f g(x)|^2 d\mu(x) = \int_X |f(x)|^2 |g(x)|^2 d\mu(x) \leq \|f\|_\infty^2 \|g\|^2 .$$

It implies $\|M_f\| \leq \|f\|_\infty$.

On the other hand, if $0 < \alpha < \|f\|_\infty$, then $\mu\{x \,|\, |f(x)| > \alpha\} > 0$ and so there is a set $Y \subset \{x \,|\, |f(x)| > \alpha\}$ of finite nonzero measure. Then, for the characteristic function χ_Y, of the set Y

$$\|M_f(\chi_Y)\|^2 = \int_Y |f(x)|^2 d\mu(x) > \alpha^2 \mu(Y) = \alpha^2 \|\chi_Y\|^2 .$$

Therefore, $\|M_f\| \geq \alpha$. In summary,

$$\|M_f\| = \|f_\infty\| .$$

We have $M_f^* = M_{\overline{f}}$ and so $M_f^* M_f = M_f M_f^* = M_{|f|^2}$. Notably, M_f is normal.

Deep spectral theorem says that all normal operators arise in this way !

- M_f is self-adjoint $\iff$ f is real:

$$\|M_f - M_f^*\| = \|M_{f-\bar{f}}\| = \|f - \bar{f}\|_\infty .$$

- M_f is positive $\iff$ $f \geq 0$ a.e. :

$$(M_f g, g) = \int_X f(x)\, |g(x)|^2 d\mu(x) ,$$

and so M_f is positive if, and only if, $\int_Y f(x) d\mu(x) \geq 0$ for each measurable set Y, which is equivalent to $f \geq 0$ a.e.

- M_f is unitary $\iff$ $|f| = 1$ a.e. :

$$\|I - M_f M_f^*\| = \|M_{(1-|f|^2)}\| = \|1 - |f|^2\|_\infty .$$

• M_f is a projection $\Longleftrightarrow$ $f = \chi_Y$ for some measurable Y:
f is real and

$$\|M_f - M_f^2\| = \|M_{f-f^2}\| = \|f - f^2\|_\infty\,.$$

Consequently, f is 0 or 1 a.e.

Note that M_f may have no eigenvalue: Consider $L^\infty[0,1]$ and $f(x) = x$. Let $\lambda \in \mathbb{C}$.

$xg(x) = \lambda g(x)$ for almost all x implies $(x - \lambda)g(x) = 0$ and so $g = 0\,a.e.$.

Matrix point of view: If H is a Hilbert space with an orthonormal basis (e_n), then $T \in B(H)$ is determined by an infinite matrix

$$((Te_m, e_n))_{m,n}\,.$$

T^* corresponds to adjoint matrix. In Example 1.4 we have seen T whose matrix is diagonal.

1.6. Example. Let $x, y \in H$ be distinct. Define

$$T_{x,y}(z) = (z, x)\, y \quad z \in H\,.$$

For $u, v \in H$ we obtain

$$(T_{x,y}u, v) = (u, x)(y, v) = (u, (v, y)x) = (u, T_{y,x}(v))\,,$$

implying

$$T_{x,y}^* = T_{y,x}\,.$$

Now

$$T_{x,y}T_{x,y}^*(z) = (x, x)(z, y)y = \|x\|^2 T_{y,y}(z)\,.$$

By exchanging the roles of x and y, we obtain

$$T_{x,y}^* T_{x,y} = \|y\|^2 T_{x,x}\,.$$

(If x and y are unit vectors, then the map $T_{x,y}$ is called a partial isometry exchanging one dimensional projections onto span of x and span of y, respectively. This is important for the structure theory of projections – we have to deal with non-normal operators!)

1.7. Proposition. *Any projection $P \in B(H)$ is an orthogonal projection of H onto $P(H)$.*

Proof: Put $M = P(H)$. M is closed because $P^2 = P$ implies that $P(H) = \{x \in H \mid Px = x\}$. We can write $H = M \oplus M^{\perp}$.
If $y \in M^{\perp}$, then $(x, Py) = (Px, y) = 0$ for each $x \in H$ and so $Py = 0$. Therefore, if $z = x + y$, where $x \in M$ and $y \in M^{\perp}$, then

$$P(x + y) = x\,.$$

2 Spectral Theory of Normal Operators

Recall that for $T \in B(H)$ the spectrum, $\mathrm{Sp}\,T$, is a subset of $\mathbb{C}$ defined by

$$\lambda \in \mathrm{Sp}\,T \Longleftrightarrow (T - \lambda I) \text{ has not an inverse in } B(H)\,.$$

Point spectrum

$$\mathrm{Sp}_p\,T = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ is not one-to one}\,.\}$$

In other words, for each $\lambda \in \sigma_p(T)$ there is a nonzero y in H such that

$$Ty = \lambda\, y\,.$$

Vector y is called an eigenvector.

Spectral radius $r(T) = \sup\{|\lambda| \mid \lambda \in \mathrm{Sp}(T)\}$.

Spectrum is always a compact subset of $\mathbb{C}$.

2.1. Proposition. *Let $T \in B(H)$ be normal. Then the following statements hold:*

(i) *If $Tx = \lambda\, x$ for some $\lambda \in \mathbb{C}$ and $x \in H$, then $T^*x = \overline{\lambda}x$.*

(ii) *If $\lambda_1 \neq \lambda_2$ are complex numbers, then*

$$\mathrm{Ker}(T - \lambda_1 I) \perp \mathrm{Ker}(T - \lambda_2 I)\,.$$

Proof (i) By normality of T, for each $x \in H$,

$$\|(T-\lambda I)x\| = \|(T-\lambda I)^* x\| = \|(T^* - \bar{\lambda}\, I)x\|\,.$$

It implies (i).
(ii) Suppose that $x, y \in H$ and $\lambda_1 \neq \lambda_2$ are in $\mathbb{C}$ such that $Tx = \lambda_1\, x$, $T\,y = \lambda_2\, y$. Then

$$\lambda_1(x,y) = (Tx,y) = (x,T^*y) = (x,\overline{\lambda_2 y}) = \lambda_2(x,y)\,.$$

Since $\lambda_1 \neq \lambda_2$, $(x,y) = 0$. □

Spectrum of a normal operator has a simpler structure than in general case.

2.2. Proposition. *Let $T \in B(H)$ be a normal operator. Then*

$$\lambda \notin \operatorname{Sp} T \iff \text{there is } c > 0 \text{ such that } \|(T-\lambda I)x\| > c\|x\| \text{ for all } x \in H\,. \tag{1}$$

Proof: Without loss of generality assume that $\lambda = 0$. Suppose that there is $c > 0$ satisfying the condition (1). Then T is one-to-one. It follows from (1) that range $T(H)$ is complete and thereby closed in H. It remains to prove that $R(T) = H$. Choose $x \in R(T)^{\perp}$. Then

$$0 = (x, TT^*x) = (x, T^*Tx) = (Tx, Tx) = \|Tx\|^2 \geq c^2\|x\|^2\,.$$

In other words, $x = 0$ and $R(T) = H$. So (1) implies $0 \notin \operatorname{Sp} T$. The reverse implication is clear. □

2.3. Corollary. *If $T \in B(H)$ is normal and $\lambda \in \operatorname{Sp} T \setminus \sigma_p(T)$, then $(T-\lambda I)(H)$ is not closed.*

Proof: If $T - \lambda I$ is one-to-one and $(T-\lambda I)(H)$ is closed, then, by the Inverse Mapping Theorem, there is a continuous linear map $S : (T-\lambda I)(H) \to H$ such that $S(T-\lambda I)x = x$ for all $x \in H$. It means that $\|x\| \leq \|S\|\|(T-\lambda I)x\|$. As $\|S\| \neq 0$, we see that

$$\|(T-\lambda I)x\| \geq \frac{1}{\|S\|}\|x\|\,.$$

In view of Proposition 2.2, $\lambda \notin \operatorname{Sp} T$. □

2.4. Corollary. *(Approximate Spectrum) If $T \in B(H)$ is normal, then $\lambda \in \operatorname{Sp} T$ if, and only if, there is a sequence (x_n) of unit vectors such that $\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$.*

Proof: By Proposition 2.2 $\lambda \in \operatorname{Sp} T \iff \inf_{\|x\|=1} \|(T - \lambda I)x\| = 0.$ □

Spectrum of a normal operator is equal to approximate point spectrum.

2.5. Corollary. *If $T \in B(H)$ is normal, then*

$$\operatorname{Sp} T \subset \overline{\{(Tx, x) \mid \|x\| = 1\}}\,.$$

Proof: If $\lambda \in \operatorname{Sp} T$ then there is a sequence (x_n) of unit vectors such that

$$\|T\,x_n - \lambda x_n\| \to 0\,.$$

It implies

$$(Tx_n - \lambda x_n, x_n) \to 0$$
$$(Tx_n, x_n) \to \lambda\,.$$

□

2.6. Theorem. *If $T \in B(H)$ is a normal operator, then the following statements hold:*

(i) *T is self-adjoint if, and only if,* $\operatorname{Sp} T \subset \mathbb{R}$.

(ii) *T is positive if, and only if,* $\operatorname{Sp} T \subset \mathbb{R}^+$.

(iii) *T is unitary if, and only if,* $\operatorname{Sp} T \subset \{z \in \mathbb{C} \mid |z| = 1\}$.

(iv) *T is a projection if, and only if,* $\operatorname{Sp} T \subset \{0, 1\}$.

Proof: We shall prove the implications $\Longrightarrow$ (the reverse implications are more complicated). In case of (i) and (ii), these implications follow from Corollary 2.5.

Suppose that T is unitary. Then $\|T\| = 1$ and so $|(Tx, x)| \leq \|x\|^2 = 1$ for all unit vectors x. By Proposition 2.5 $\operatorname{Sp} T$ is a subset of the unit disc. If $\lambda \in \operatorname{Sp} T$, then λ is nonzero and $\frac{1}{\lambda} \in \operatorname{Sp} T^{-1} = \operatorname{Sp} T^*$. As T^* is unitary we have that $|\frac{1}{\lambda}| \leq 1$. Hence, $|\lambda| = 1$.
(iv) is a consequence of proposition 1.7

□

2.7. Proposition. *(C^*-property) If $T \in B(H)$, then*

$$\|T^*T\| = \|T\|^2 .$$

Proof: First observe that $\|T\| = \|T^*\|$ (consider e.g. corresponding bilinear forms). For $x \in H$ we have

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*T\| \cdot \|x\|^2 .$$

Hence,

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\|^2 .$$

□

2.8. Proposition. *If $T \in B(H)$ is normal, then*

$$r(T) = \|T\| .$$

Proof: First suppose that T is self-adjoint. Then by the C^*-property

$$\|T\|^2 = \|T^*T\| = \|T^2\| .$$

If T is normal, then

$$\|T^2\|^2 = \|(T^2)^*T^2\| = \|(T^*T)^2\| = \|T^*T\|^2 = \|T\|^4 .$$

(We have used the fact that T^*T is self-adjoint.) Consequently,

$$\|T^2\| = \|T\|^2 ,$$

and in turn $\|T^{2^n}\| = \|T\|^{2^n}$ for all n. By the spectral radius formula

$$r(T) = \lim_n \|T^n\|^{1/n} = \lim_n \|T^{2^n}\|^{1/2^n} = \lim_n \|T\| = \|T\| .$$

□

Very useful concept in operator theory is that of *numerical range* of an operator $T \in B(H)$:

$$N(T) = \overline{\{(Tx, x) \mid \|x\| = 1\}} .$$

By a *numerical radius* of T we mean

$$n(T) = \sup_{\|x\|=1} |(Tx, x)| .$$

It is clear that, in general, $n(T) \leq \|T\|$.

2.9. Proposition. *Let $T \in B(H)$. Then the following statements hold*

(i) *If T is normal, then*

$$\|T\| = r(T) = n(T)\,.$$

(ii) *If T is self-adjoint, then $\|T\|$ or $-\|T\|$ is in $\operatorname{Sp} T$.*

Proof: (i) If T is normal, then $\operatorname{Sp} T \subset N(T)$ by Proposition 2.5. Obviously,

$$r(T) \le n(T) \le \|T\| = r(T)$$

and so $r(T) = n(T)$.
(ii) By working with $\|T\|^{-1}T$ in place of T, we can assume that $\|T\| = 1$. Then there is a sequence of unit vectors (x_n) such that $\|Tx_n\| \to 1$. Thanks to this

$$\|(I - T^2)\,x_n\|^2 = \|x_n\|^2 + \|T^2 x_n\|^2 - 2\Re(T^2 x_n, x_n) \le 2 - 2\|T\,x_n\|^2 \to 0$$

as $n \to \infty$. We see that $1 \in \operatorname{Sp} T^2$. It means that $T + I$ or $T - I$ has no inverse, for otherwise,

$$T^2 - I = (T + I)(T - I)$$

would have an inverse, which is not possible. □

In view of the previous result we can say that the norm of a normal operator is given by the extreme of the corresponding quadratic form.

3 Algebraic Aspects and Applications

The facts mentioned below follow from Exercises. The set of normal operators is stable under forming powers and scalar multiples. If T is normal, then the smallest $*$-subalgebra of $B(H)$ containing T is commutative. Any operator $T \in B(H)$ can be written as $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint. Moreover, any self adjoint operator is a difference of two positive operators. If T is self-adjoint, then T^2 is always positive. The converse also holds. (The proof is more complicated and will be omitted.)

3.1. Proposition. *For $T \in B(H)$ the following conditions are equivalent:*

(i) *T is positive*

(ii) *$T = A^*A$ for some $A \in B(H)$.*

(iii) *$T = S^2$ for some self-adjoint $S \in B(H)$. (S is denoted by $T^{1/2}$ and called the square root of T).*

If T is self-adjoint, then e^{iT} is unitary. The converse also holds:

3.2. Proposition. *For any unitary operator $U \in B(H)$ there is a self-adjoint operator $T \in B(H)$ with $\|T\| \le 2\pi$ such that $U = e^{iT}$.*

In Physics: $t \in \mathbb{R} \to e^{itH}$, where H is Hamiltonian (Energy), describes time development of the system (solution of the Schrödinger equation).

Another important example of a unitary map is the Fourier-Plancherel transform:

$$f \in L^2(\mathbb{R}) \to \hat{f}(\omega) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} dt\,.$$

4 Compact Operators

<u>Notation:</u>
B – Banach space
For $x \in B$ and $\varepsilon > 0$ denote $B_\varepsilon(x) = \{y \mid \|x - y\| \le \varepsilon\}$
$B_1 = B_1(0)$.

4.1. Definition. A set X in a Banach space B is said to be *compact* if for each system U of open subsets of B with $X \subset \cup_{O \in U} O$ there is a finite subset $U' \subset U$ with $X \subset \cup_{O \in U'} O$. A set $X \subset B$ is said to be *relatively compact* if its closure, $\overline{X}$, is compact.

Related concept to compactness is total boundedness.

4.2. Definition. A set X in a Banach space B is said to be *totally bounded* if for each $\varepsilon > 0$ there exist $x_1, \dots, x_n \in X$ such that

$$X \subset \cup_{i=1}^{n} B_\varepsilon(x_i)\,.$$

4.3. Theorem. *$X \subset B$ is compact if, and only if, X is closed and totally bounded. $X \subset B$ is relatively compact if, and only if, X is totally bounded.*

Basic facts about compact sets:

- If $X \subset B$ is relatively compact, then for each sequence $(x_n) \subset X$ there is a cauchy subsequence (x_{n_k}).
- Any relatively compact set is bounded.
- Any bounded set in a finite-dimensional space is relatively compact.
- Unit ball B_1 is compact if, and only if, $\dim B < \infty$.
- Let $f : B_1 \to B_2$ be a continuous map between Banach spaces. If $X \subset B_1$ is (relatively) compact, then the image $f(X)$ is (relatively) compact in B_2.

4.4. Definition. A linear operator $T : F \to G$ between Banach spaces F and G is called *compact* if

$$T(F_1) \text{ is relatively compact}\,.$$

Basic facts about compact operators:

$T : F \to G$ is a linear map between Banach spaces.

- T is compact if it maps bounded sets to relatively compact sets. In particular, compact maps are continuous.
- If T is compact, then for each bounded sequence $(x_n) \subset F$ there is a subsequence (x_{n_k}) such that (Tx_{n_k}) is convergent.
- The identity map on a Banach space B is compact if, and only if, $\dim B < \infty$.
- Any bounded operator with finite-dimensional range is compact.

4.5. Corollary. *If $T : B \to B$ is a compact map, then for each nonzero $\lambda \in \mathbb{C}$*

$$\dim \operatorname{Ker}(T - \lambda I) < \infty$$

Proof: T restricted to $\mathrm{Ker}(T-\lambda I)$ is a nonzero multiple of I. Therefore $T : \mathrm{Ker}(T-\lambda I) \to \mathrm{Ker}(T-\lambda I)$ is compact if, and only if, $\dim \mathrm{Ker}(T-\lambda I) < \infty$. □

Compact operators on Banach spaces have special spectral properties.

4.6. Theorem. *Let $T : B \to B$ be a compact operator. Then $\mathrm{Sp}\, T$ is countable, and each nonzero point of $\mathrm{Sp}\, T$ is an eigenvalue and an isolated point of $\mathrm{Sp}\, T$. For each nonzero $\lambda \in \mathrm{Sp}\, T$, the space $\mathrm{Ker}(T-\lambda I)$ has finite dimension.*

We shall prove this theorem for normal compact operators on Hilbert spaces later.

Notation:

$K(H)$ ... compact operators acting on a Hilbert space H

$F(H)$ ... finite rank operators acting on a Hilbert space H.
($T \in F(H)$ if, and only if, $\dim T(H) < \infty$.)

These classes of operators form a special structure in $B(H)$

4.7. Definition. An *ideal* $J \subset B(H)$ is a linear subspace of $B(H)$ such that

$$ST, \qquad TS \in J$$

whenever $T \in J$ and $S \in B(H)$.

4.8. Proposition. *(i) $F(H) \subset K(H)$ and each $T \in F(H)$ is a linear combination of the operators of the form*

$$T_{x,y}(z) = (z, x)\, y\,,$$

where $x, y \in H$.

(ii) $F(H)$ and $K(H)$ are ideals in $B(H)$. Moreover, $K(H)$ is a closed ideal in $B(H)$.

Proof: (i) $F(H) \subset K(H)$ because any finite-dimensional operator is compact.

Take $T \in F(H)$ and let P be the projection of H onto $R(T)$. Then

$$P = P_1 + P_2 + \cdots + P_n\,,$$

where each P_i is a one-dimensional projection. As

$$T = PT$$

the problem of description of T reduces to a rank one operator: $\dim R(T) = 1$. Suppose $R(T) = span\{y\}$, where $\|y\| = 1$. Then, for each $z \in H$,

$$Tz = (Tz, y)y = (z, T^*y)y$$

and so $T = T_{T^*y,y}$.

(ii) $K(H)$ is a subspace of $B(H)$ because the sum of finitely many totally bounded sets is totally bounded and scalar multiple of a totally bounded set is totally bounded as well. If $S \in B(H)$, then

$T \in F(H) \Longrightarrow ST, TS \in F(H)$ (linear algebra)

$T \in K(H) \Longrightarrow ST, TS \in K(H)$.

(Last implication is due to the fact that bounded operators map relatively compact sets to relatively compact sets and that continuous image of a relatively compact set is relatively compact.)

Closedness of $K(H)$:

Suppose that $(T_n) \subset K(H)$, $T_n \to T \in B(H)$. Given $\varepsilon > 0$ there is n_0 such that $\|T - T_n\| < \varepsilon/3$ whenever $n \geq n_0$. There are $x_1, \ldots, x_k \in H_1$ such that

$$T_{n_0}(H_1) \subset \bigcup_{i=1}^{k} B(T_{n_0}x_i, \frac{\varepsilon}{3})\,.$$

Now for any $x \in H_1$, $Tx \in T(H_1)$ and

$$\|Tx - T_{n_0}x\| \leq \varepsilon/3\,.$$

There is $1 \le j \le k$ such that

$$\|T_{n_0}x - T_{n_0}x_j\| \le \varepsilon/3$$

and so

$$\begin{aligned}\|Tx - Tx_j\| &\le \|Tx - T_{n_0}x\| + \|T_{n_0}x - T_{n_0}x_j\| + \|T_{n_0}x_j - Tx_j\| \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Hence, $T(H_1)$ is totally bounded. □

4.9. Example. Suppose that $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of H and $T \in B(H)$ is defined by $Te_n = \frac{1}{n}e_n$. (T is a diagonal operator.) Set

$$T_N x = \sum_{n=1}^{N} \frac{1}{n}(x, e_n)e_n.$$

Then

$$\|(T - T_N)x\|^2 = \| \sum_{n=N+1}^{\infty} \frac{1}{n}(x, e_n)e_n\|^2 = \sum_{n=N+1}^{\infty} \frac{1}{n^2}|(x, e_n)|^2 \le \frac{1}{(N+1)^2}\|x\|^2.$$

Therefore $\|T - T_N\| \to 0$ as $N \to \infty$. By the previous result T is compact. (Observe that the same is true whenever $Te_n = \lambda_n e_n$, where $\lambda_n \to 0$.)

$c_0 \hookrightarrow K(H)$ – noncommutative c_0.

We shall now develop theory of self-adjoint and normal compact operators.

4.10. Proposition. *If $T \in K(H)$ is self-adjoint, then $\|T\|$ or $-\|T\|$ must be an eigenvalue of T.*

Proof: (We know that $\|T\|$ or $-\|T\|$ is in $\operatorname{Sp} T$.)
Without loss of generality assume that $\|T\| = 1$. Then

$$1 = \|T\| = \sup_{\|x\|=1} |(Tx, x)|.$$

As (Tx, x) is real for all x, there exists a sequence (x_n) of unit vectors such that

$$(Tx_n, x_n) \to 1(\text{ or } \to -1 \text{ which is the same})\,.$$

Using compactness we can pass to a subsequence of (x_n), denoted by the same symbol, such that

$$Tx_n \to x \in H_1\,.$$

Then

$$(x, x_n) \to 1 \text{ and so } x_n \to x\,.$$

Now $x_n \to x, Tx_n \to x$ implies $Tx = x$. □

4.11. Proposition. *Let $T \in K(H)$ and $(e_n)_{n=1}^{\infty}$ be an orthonormal sequence in H. Then*

$$Te_n \to 0 \text{ as } n \to \infty\,.$$

Proof: Without loss of generality we can assume that

$$\lim_n Te_n = x\,.$$

Suppose that $x \neq 0$ and try to reach a contradiction. Given n we can find $k(n)$ such that

$$\|Te_m - x\| \leq \frac{1}{\sqrt{n}}$$

for all $m \geq k(n)$. Now set

$$u_n = \frac{1}{\sqrt{n}}(e_{k(n)} + e_{k(n)+1} + \cdots + e_{k(n)+n-1}).$$

Then $\|u_n\| = 1$. As

$$\|\sum_{i=k(n)}^{k(n)+n-1} Te_i\| \geq \|nx\| - \sum_{i=k(n)}^{k(n)+n-1} \|Te_i - x\| \geq n\|x\| - n\frac{1}{\sqrt{n}}$$

we obtain

$$\begin{aligned} \|Tu_n\| &\geq \frac{1}{\sqrt{n}}\Big[n \cdot \Big(\|x\| - \frac{1}{\sqrt{n}}\Big)\Big] \\ &= \sqrt{n}\|x\| - 1 \to \infty \text{ as } n \to \infty\,. \end{aligned}$$

So T is unbounded - a contradiction. □

4.12. Theorem. *(Spectral theorem for normal compact operators) Let H be a separable Hilbert space of infinite dimension and T a normal compact operator acting on H. Then there is an orthonormal basis $(e_n)_{n=1}^{\infty}$ of H and a sequence of complex numbers $\lambda_n \to 0$ such that*

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n \tag{2}$$

for all $x \in H$.

Proof: We show first that T is diagonalizable. By Zorn's lemma there is a maximal orthonormal set E of eigenvectors of T. If L is the closed linear span of E, then $H = L \oplus L^{\perp}$. Observe that $L^{\perp}$ is T-invariant. For this fix $x \in L^{\perp}$ and take arbitrary $y \in E$. Then there is a scalar λ such that $Ty = \lambda y$. It gives

$$(y, Tx) = (T^* y, x) = (\overline{\lambda} y, x) = 0\,.$$

Therefore, T restricts to a compact normal operator acting on $L^{\perp}$. We are going to show that $L^{\perp} = \{0\}$.

First we show that any nonzero point λ in the spectrum of T is an eingenvalue. By Corollary 2.4 there is a sequence (x_n) in H_1 such that

$$Tx_n - \lambda x_n \to 0$$

as $n \to \infty$. As T is compact we can, by passing to a subsequence, assume that

$$\lim_n Tx_n = y\,.$$

Then $\lambda x_n \to y$ and so $x_n \to \frac{y}{\lambda}$. In turn, $y \neq 0$,

$$y = \lim_n Tx_n = \frac{Ty}{\lambda}\,,$$

saying that

$$Ty = \lambda y\,.$$

Now, if $L^{\perp}$ were nonzero, then T would have a nonzero eigenvector in $L^{\perp}$, which is excluded by maximality of E. Hence $L^{\perp} = \{0\}$.

Summing it up, E is an orthonormal basis of H and so that T is diagonalizable. In other words, T is of the form (2) for some sequence (λ_n). That $\lambda_n \to 0$ follows from Proposition 4.11. □

5 Trace Class and Hilbert-Schmidt Operators

- Applications to integral equations, Gaussian stochastic processes, unitary representations of locally compact groups, ...
- quantization of ℓ^1, ℓ^2.

5.1. Definition. Let $T \in B(H)$ be a positive operator and $(e_n)_{n=1}^{\infty}$ an orthonormal basis of H. Define

$$\operatorname{trace} T = \sum_{n=1}^{\infty} (Te_n, e_n)\,.$$

(It may happen that $\operatorname{trace} T = \infty$.)

Remarks: In the matrix representation $\operatorname{trace} T$ is a sum of diagonal elements.

5.2. Proposition. (i) *For a given positive $T \in B(H)$, $\operatorname{trace} T$ does not depend on the choice of an orthonormal basis (e_n).*

(ii)

$$\begin{aligned} \operatorname{trace}(T_1 + T_2) &= \operatorname{trace} T_1 + \operatorname{trace} T_2 \\ \operatorname{trace}(\lambda T_1) &= \lambda \operatorname{trace} T_1\,, \end{aligned}$$

whenever $T_1, T_2 \geq 0$ and $\lambda \geq 0$.

Proof: (i) Fix two orthonormal bases (e_k) and (f_k) and $T \in B(H)$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} (Te_k, e_k) &= \sum_{l,k=1}^{\infty} (Te_k, f_l)\overline{(e_k, f_l)} = \sum_{l,k=1}^{\infty} (e_k, Tf_l)\overline{(e_k, f_l)} \\ &= \sum_{l,k=1}^{\infty} \overline{(Tf_l, e_k)}(f_l, e_k) = \sum_{l=1}^{\infty} (f_l, Tf_l) \\ &= \sum_{l=1}^{\infty} (Tf_l, f_l)\,. \end{aligned}$$

(ii) obvious

5.3. Corollary. $\operatorname{trace}(U^*TU) = \operatorname{trace} T$ *whenever* U *is unitary and* $T \geq 0$.

5.4. Example. *If* T *is a positive operator acting on an* n*-dimensional Hilbert space, then*

$$\operatorname{trace} T = \lambda_1 + \lambda_2 + \cdots + \lambda_n \,,$$

where λ_i*'s are eigenvalues of* T *(counted with multiplicity).*

5.5. Definition. (i) A positive $T \in B(H)$ is a *trace class operator* if $\operatorname{trace} T < \infty$.

(ii)

$$\mathcal{L}^1(H) = span\{T \geq 0 \mid \operatorname{trace} T < \infty\}$$

is the set of *trace class operators.*

If $T \in \mathcal{L}^1(H)$, then

$$T = P_1 - P_2 + i(P_3 - P_4) \,,$$

where $P_i \geq 0$ and $\operatorname{trace} P_i < \infty$. The decomposition is not unique, but the basic properties of the trace imply that there is a unique linear functional, denoted by trace, on $\mathcal{L}^1(H)$ defined by

$$\operatorname{trace} T = \operatorname{trace} P_1 - \operatorname{trace} P_2 + i(\operatorname{trace} P_3 - \operatorname{trace} P_4) \,.$$

Obviously, for every $T \in \mathcal{L}^1(H)$ and every orthonormal basis $e_1, e_2, \ldots$ we have

$$\operatorname{trace} T = \sum_{n=1}^{\infty} (Te_n, e_n) \,,$$

where the series on the right hand side is absolutely convergent.

5.6. Definition. An operator $T \in B(H)$ is called a *Hilbert-Schmidt operator* if

$$\operatorname{trace}(T^*T) < \infty \,.$$

$\mathcal{L}^2(H)$ ..... set of all Hilbert-Schmidt operators acting on H.

Observe that

$$T \in \mathcal{L}^2(H) \Longleftrightarrow \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$$

for any orthonormal basis $(e_n)_{n=1}^{\infty}$ of H.

5.7. Proposition. *$\mathcal{L}^2(H)$ is a self-adjoint ideal in $B(H)$.*

Proof: Let $A, B \in \mathcal{L}^2(H)$.
parallelogram law:

$$(A+B)^*(A+B) + (A-B)^*(A-B) = 2\,A^*A + 2\,B^*B\,.$$

It implies that

$$0 \le (A+B)^*(A+B) \le 2A^*A + 2B^*B\,.$$

Consequently,

$$\operatorname{trace}[(A+B)^*(A+B)] \le 2\operatorname{trace} A^*A + 2\operatorname{trace} B^*B < \infty\,,$$

and so $A+B \in \mathcal{L}^2(H)$. Hence $\mathcal{L}^2(H)$ is a subspace of $B(H)$.
We shall now prove that $\operatorname{trace} A^*A = \operatorname{trace} AA^*$ (this is of independent importance). For this fix two orthonormal basis (e_n) and (f_k) of H. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \|Ae_n\|^2 &= \sum_{n,k=1}^{\infty} |(Ae_n, f_k)|^2 \\ &= \sum_{n,k=1}^{\infty} |(e_n, A^*f_k)|^2 = \sum_{n,k=1}^{\infty} |(A^*f_k, e_n)|^2 \\ &= \sum_{k=1}^{\infty} \|A^*f_k\|^2\,.\end{aligned}$$

In other words, $\mathcal{L}^2(H)$ is self-adjoint. If $B \in B(H)$ and $A \in \mathcal{L}^2(H)$, then

$$\sum_{n=1}^{\infty} \|BAe_n\|^2 \le \|B\|^2 \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty\,.$$

So $\mathcal{L}^2(H)$ is a left ideal and by self-adjointeness it is an ideal.

5.8. Proposition.

$$\mathcal{L}^2(H) \subset K(H)\,.$$

Proof: $F(H) \subset K(H)$. For any $x \in H_1$ and any $T \in B(H)$

$$\|Tx\|^2 \leq \sum_{n=1}^{\infty} \|Te_n\|^2 = \operatorname{trace} T^*T\,,$$

where $(e_n)_{n=1}^{\infty}$ is an orthonormal basis containing x. This means that

$$\|T\|^2 \leq \operatorname{trace} T^*T\,.$$

Suppose now that $T \in \mathcal{L}^2(H)$. Fix an orthonormal basis $(e_n)_{n=1}^{\infty}$. Let P_N be the orthogonal projection onto $span\{e_1, \ldots, e_N\}$. Put $F_N = TP_N$. Then

$$\|T-F_N\|^2 \leq \operatorname{trace}((I-P_N)T^*T(I-P_N)) = \sum_{n=N+1}^{\infty} \|Te_n\|^2 \to \infty \text{ for } N \to \infty\,.$$

Therefore $T \in K(H)$. □

5.9. Corollary. *A normal Hilbert-Schmidt operator T is diagonalizable and for its sequence (λ_n) of eigenvalues we have*

$$\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty\,.$$

A natural inner product can be introduced on $\mathcal{L}^2(H)$.

For $A, B \in \mathcal{L}^2(H)$ define

$$(A, B)_2 \;\; = \;\; \sum_{n=1}^{\infty} (B^*Ae_n, e_n) = \sum_{n=1}^{\infty} (Ae_n, Be_n)\,.$$

$$(\sum_{n=1}^{\infty} |(Ae_n, Be_n)| \leq \sum_{n=1}^{\infty} \|(Ae_n\|^2 \sum_{n=1}^{\infty} \|(Be_n\|^2 < \infty\,.)$$

Note that $\|A\|_2 = \operatorname{trace} A^*A$.
It can be proved that $(\mathcal{L}^2(H), (\cdot,\cdot)_2)$ is a Hilbert space.

Hilbert-Schmidt integral operator:

(X, μ) .... σ-finite measure space.

$$k(x, y) \in \mathcal{H} = L^2(X \times X, \mu \times \mu)\,.$$

Define an operator T on $L^2(X, \mu)$ by

$$T\xi(x) = \int_X k(x, y)\xi(y)d\mu(y)\,. \tag{3}$$

This definition is correct because $k(x, \cdot) \in L^2(X, \mu)$ for a.a. $x \in X$.

Another application of Fubini's theorem implies that for $\xi, \nu \in L^2(X, \mu)$

$$\int_X |T\xi(x)||\nu(x)|d\mu(x) \leq \int_{X\times X} |k(x, y)||\xi(x)||\nu(x)|d\mu(x)d\mu(y) \leq \|k\|\|\xi\|\,\|\nu\|$$

Hence $T \in B(L^2(X, \mu))$ and $\|T\| \leq \|k\|$.

Let us now compute the trace of T^*T. Choose an orthonormal basis $(e_n)_{n=1}^{\infty}$ of $L^2(X, \mu)$. Then

$$u_{mn}(x, y) = e_n(x)\overline{e_m(y)} \qquad m, n = 1, 2, \dots\,.$$

form an orthonormal basis of $L^2(X \times X, \mu \times \mu)$. We observe

$$\begin{aligned} (Te_m, e_n) &= \int_X Te_m(x)\overline{e_n(x)}d\mu(x) = \int_{X\times X} k(x, y)\overline{e_n(x)}e_m(y)d\mu(y)d\mu(x) \\ &= (k, u_{mn})\,. \end{aligned}$$

In turn,

$$\operatorname{trace} T^*T = \sum_{m,n=1}^{\infty} |(Te_m, e_n)|^2 = \sum_{m,n=1}^{\infty} |(k, u_{mn})|^2 = \|k\|^2\,.$$

We summarize the results of this discussion in the following proposition:

5.10. Proposition. *Let* (X, μ) *be a* σ*-finite measure space. For every function* $k \in L^2(X \times X, \mu \times \mu)$ *there is a unique bounded operator* T_k *on* $L^2(X, \mu)$ *satisfying*

$$T\xi(x) = \int_X k(x,y)\xi(y)d\mu(y) \qquad \xi \in L^2(X, \mu)\,.$$

Then T_k *is a Hilbert-Schmidt operator with the Hilbert-Schmidt norm* $\|k\|$.