Graph Theory

Euler's formula for planar graphs

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Euler's formula

Let G be a connected planar simple graph with e edges and v vertices. Also, let f be the number of regions in a plane graph corresponding to G. Then $f - e + v = 2$.

proof by induction on number of edges while fixing n

A corollary to Euler's formula

If a connected planar simple graph has e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

$2e = \sum_{r \in f}$ number of edges bounding r
every face, including the outer face is bounded with at least 3 edges: hence, $2e \geq 3f$

special case: 2 edges bound a face whenever there is a spanning path on three vertices

Hence, K_5 is non-planar.

Another corollary to Euler's formula

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

now $2e \geq 4f$

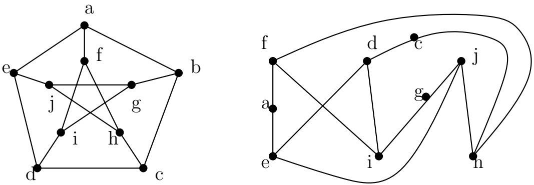
Hence, $K_{3,3}$ is non-planar.

Yet another corollary to Euler's formula

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

if $\delta(G) \geq 6$, then $\sum_v deg(v) = 2e$ contradicts $e \leq 3v - 6$

Kuratowski's theorem



a subgraph of Petersen graph (shown right) is homeomorphic to $K_{3,3}$

A graph is *nonplanar* if and only if it contains a subgraph *homeomorphic* to $K_{3,3}$ or K_5.

Let G(V, E) be a graph. An *elementary subdivision* on G involves obtaining another graph G' by removing an edge $e = (u, v) \in E$ and adding a new vertex w together with edges (u, w) and (w, v). A *series reduction* operation is precisely the inverse transformation of elementary subdivison that is applied to vertices of degree two.

Two graphs G' and G'' are said to be *homemorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions. Equivalently, G' and G'' are homeomorphic if they are isomorphic or can be reduced to isomorophic graphs by a sequence of series reductions.

— not proved in class

Observation

- G is planar iff all minors of G are planar. Hence, the family of simple planar graphs is minor closed.

Wagner's theorem

A finite graph is planar iff it does not have K_5 or $K_{3,3}$ as a minor. (That is, $\{K_5, K_{3,3}\}$ is the obstruction set of the family of planar graphs.)[1]

— not proved in class

[1] The celebrated *Robertson & Seymour theorem*: Every minor closed graph family has a finite obstruction set.

Drawing planar graphs

- Let G be planar and let π be a plane drawing of G. Also, let F be an inner face of π. Then there exists a plane drawing π' of G that has the verices of F defining the outer face of π'.

  rotate the sphere so that the face that correspond to F in the stereographic projection σ_N of G becomes north pole N before projecting back with σ_N^{-1}

- The *skeleton* of a convex polytope P is planar.

  for a point p interior to P and a sphere S with center p so that S contains P, choose a face of the central projection of P onto S, say π, as north pole N and stereographically project π onto the plane with w.r.t. N

- *Wagner '36; Fary '48; Stein '51*: Every planar graph can be drawn with line segments. — not proved

- *Koebe '36*: Every planar graph can be represented as the contract graph of disks. — not proved

Outline

Number of regular polyhedra

There are only five regular polyhedra. [2]

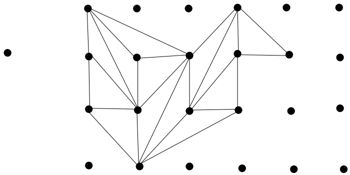
- every regular polyhedra is convex; hence, has a planar embedding
- noting that $pf = 2e$, $qv = 2e$: $\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$
- hence, the only combinations possible are:

$\{3,3\}$: $e = 6, f = 4, v = 4$ (tetrahedron)
$\{3,4\}$: $e = 12, f = 8, v = 6$ (octahedron)
$\{3,5\}$: $e = 30, f = 20, v = 12$ (icosahedron)
$\{4,3\}$: $e = 12, f = 6, v = 8$ (cube)
$\{5,3\}$: $e = 30, f = 12, v = 20$ (dodecahedron)

every other p and q combination yield a meaningless value for e

[2] A polygon that is both equilateral and equviangular is called *regular*. A (convex) polyhedron is said to be a *regular polyhedron* if all its faces are equal regular polygons and the same number of faces meet at a vertex. A regular polyhderon having p sided regular polygons as faces with q faces meeting at every vertex is denoted by $\{p, q\}$ polyhedron.

Pick's theorem



The area $A(Q)$ of any (not necessarily convex) polygon $Q \subseteq R^2$ with integral vertices is given by $A(Q) = n_{int} + \frac{1}{2}n_{bd} - 1$ where n_{int} and n_{bd} are the number of integral points in the interior and on the boundary of Q respectively.

- area of every elementary triangle[3] with the vertices from a unit grid has area $\frac{1}{2}$
- triangulate the Q with n_{int} and n_{bd} such that every triangle is elementary:
  $A(Q) = \frac{1}{2}(f-1)$
  further, $3(f-1) = 2e_{int} + e_{bd}$ i.e., $f = 2(e-f) - e_{bd} + 3$
  and $e_{bd} = n_{bd}$

[3] a convex polygon is *elementary* if its vertices are from the lattice and the polygon does not contain any further lattice points

Crossing number

Let $G(V,E)$ be a connected simple graph. The *crossing number* of G, $cr(G)$, is the smallest number of crossings among all drawings of G[4], where crossings of more than two edges in one point are not allowed. Then $cr(G) \geq m - 3n + 6$.

while treating the crossings as nodes with edges defined appropriately,

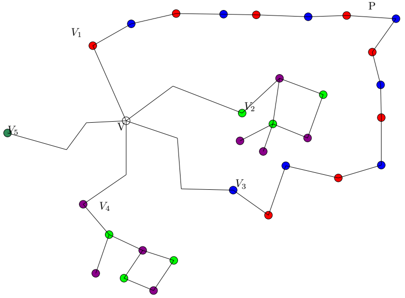
$m + 2cr(G) \leq 3(n + cr(G)) - 6$

[4] note that in such a minimal drawing, the following situations are ruled out: (i) no edge can cross itself; (ii) edges with a common endvertex cannot cross; (iii) no two edges cross twice

Five color theorem

Every planar graph is 5-colorable.[5]

inductive step: include a vertex whose degree does not exceed five



v_2 and v_4 lie in diferent faces of cycle

for a vertex v of degree five, let $H = G - v$; let H_{13} (resp. H_{24}) be the subgraph of H induced by vertices colored 1 or 3 (resp. 2 or 4)

either v_1, v_3 belong to distinct components of H_{13}

or, v_2, v_4 belong to distinct components of H_{24}

[5] in fact, *Appel and Haken '76* proved that every planar graph is four colorable but the proof has close to 2000 cases and several of those are proved using computer simulations; on the other hand, *Grotzsch's theorem* states that every planar graph not containing a triangle is 3-colorable